EURASIAN JOURNAL OF ACADEMIC RESEARCH
Innovative Academy Research Support Center
UIF = 8.1 | SJIF = 5.685
www.in-academy.uz

BOUNDARY VALUE PROBLEM FOR THE EQUATION OF THE TRANSVERSE VIBRATION OF A BEAM

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https://www.doi.org/10.5281/zenodo.7854545

ARTICLE INFO

Received: 12th April 2023
Accepted: 21th April 2023
Online: 22th April 2023

KEY WORDS

ABSTRACT

Consider the transverse vibrations of a thin beam. The main difference between beam vibrations and transverse string vibrations is that the beam resists bending. Vibrations of a beam clamped at one end are described by the following equation [3]

$$u_{tt} + a^2 u_{xxxx} = 0, \qquad (1)$$

(Here $u = u(x,t) -$ beam displacement). Boundary conditions for a given end $(x=0)$ are the immobility of the beam and the horizontality of the tangent $\left(\frac{\partial u}{\partial t}(0,t) = 0\right)$, at the free end $(x=l)$ must be zero bending moment $M = -E\frac{\partial^2 u}{\partial x^2}J$ $(E -$ modulus of elasticity of the beam material, $J -$ moment of inertia of the beam section relative to its horizontal axis) and tangential force $F = -EJ\frac{\partial^3 u}{\partial x^3}$. Note that in equation (1) $a^2 = -EJ/\rho S$ $(\rho -$ beam material density, $S -$ beam cross-sectional area).

Consider the following mixed problem:

$$\begin{cases} u_{tt} + a^2 u_{xxxx} = 0, \quad 0 < x < l, \ t > 0, & (2) \\ u(0,t) = 0, \quad u_x(0,t) = 0, & (3) \\ u_{xx}(l,t) = 0, \ u_{xxx}(l,t) = 0, \qquad t \ge 0, & \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l. & (4) \end{cases}$$

We will solve this problem by the method of separation of variables [1] under the assumption that time-periodic t beam vibrations. Assuming $u(x,t) = X(x)\cdot T(t)$ and substituting the proposed form of the solution into equation (2), we obtain [2]

$$X(x)\cdot T''(t) + a^2 X^{IV}(x)\cdot T(t) = 0.$$

from this we get following

$$\frac{T''(t)}{a^2 T(t)} = -\frac{X^{IV}(x)}{X(x)} = -\lambda.$$

It's clear that $\lambda > 0$ (for the existence of periodic t decisions). For function $X(x)$ we obtain the problem of natural oscillations

$$X^{IV}(x) - \lambda X(x) = 0 \qquad (5)$$

under boundary conditions

$$X(0)=0,\ \ X'(0)=0,\ \ X''(l)=0,\ \ X'''(x)=0. \qquad (6)$$

The general solution of equation (5) is represented as

$$X(x) = Ach\left(\sqrt[4]{\lambda}x\right) + Bsh\left(\sqrt[4]{\lambda}x\right) + C\cos\left(\sqrt[4]{\lambda}x\right) + D\left(\sqrt[4]{\lambda}x\right).$$

From conditions $X(0)=0,\ X'(0)=0$ we find that $A+C=0,\ B+D=0.$ this implies

$$X(x) = A\left[ch\left(\sqrt[4]{\lambda}x\right) - \cos\left(\sqrt[4]{\lambda}x\right)\right] + B\left[sh\left(\sqrt[4]{\lambda}x\right) - \sin\left(\sqrt[4]{\lambda}x\right)\right].$$

Boundary conditions (6) at the right end of the beam give

$$\begin{cases} A\left[ch\left(\sqrt[4]{\lambda}x\right) + \cos\left(\sqrt[4]{\lambda}x\right)\right] + B\left[sh\left(\sqrt[4]{\lambda}x\right) + \sin\left(\sqrt[4]{\lambda}x\right) = 0\right], \\ A\left[sh\left(\sqrt[4]{\lambda}x\right) - \sin\left(\sqrt[4]{\lambda}x\right)\right] + B\left[ch\left(\sqrt[4]{\lambda}x\right) + \cos\left(\sqrt[4]{\lambda}x\right)\right] = 0. \end{cases} \qquad (7)$$

Homogeneous system (with respect to unknown A And $B)$ (7) has nontrivial solutions if its determinant is equal to zero:

$$\begin{vmatrix} ch\left(\sqrt[4]{\lambda}x\right) + \cos\left(\sqrt[4]{\lambda}x\right) & sh\left(\sqrt[4]{\lambda}x\right) + \sin\left(\sqrt[4]{\lambda}x\right) \\ sh\left(\sqrt[4]{\lambda}x\right) - \sin\left(\sqrt[4]{\lambda}x\right) & ch\left(\sqrt[4]{\lambda}x\right) + \cos\left(\sqrt[4]{\lambda}x\right) \end{vmatrix} = 0. \qquad (8)$$

From equation (8) we obtain an algebraic equation for calculating the eigenvalues tasks

$$sh^2\left(\sqrt[4]{\lambda}l\right) - \sin^2\left(\sqrt[4]{\lambda}l\right) = ch^2\left(\sqrt[4]{\lambda}l\right) + 2ch^2\left(\sqrt[4]{\lambda}l\right)\cos^2\left(\sqrt[4]{\lambda}l\right) + \cos^2\left(\sqrt[4]{\lambda}l\right). \qquad (9)$$

Denoting $\mu = \sqrt[4]{\lambda}l$ and using the equality $sh^2x + 1 = ch^2x,$ from equation (9) we find

$$ch\mu\cdot\cos\mu = -1 \qquad (10)$$

This equation can be solved graphically (Fig. 1). The roots of equation (10) are

$\mu_1 = 1,875;\quad \mu_2 = 4,694;\quad \mu_3 = 7,854;\quad \mu_n \approx \frac{\pi}{2}(2n-1)$ at $n > 3.$

Further, for the function $T(t)$ we have the equation

$T''(t) + \lambda_n a^2 T(t) = 0.$

Its general solution is written as

$$T_n(t) = A_n \cos\left(a\sqrt{\lambda_n} t\right) + B_n \sin\left(a\sqrt{\lambda_n} t\right) = A_n \cos\left(a\frac{\mu_n^2}{l^2}t\right) + B_n \sin\left(a\frac{\mu_n^2}{l^2}t\right),$$

Where A_n And $B_n -$ arbitrary constants.

Consequently, the "atoms" of the solution to problem (2), (3) are formed by the following functions

$$u_n(x,t) = \left[A_n \cos\left(a\frac{\mu_n^2}{l^2}t\right) + B_n \sin\left(a\frac{\mu_n^2}{l^2}t\right)\right] \cdot X_n(x),$$

Where

$$X_n(x) = \frac{(sh\mu_n + \sin\mu_n)\left[ch\left(\frac{\mu_n}{l}x\right) - \cos\left(\frac{\mu_n}{l}x\right)\right]}{sh\mu_n + \sin\mu_n} -$$

$$-\frac{(ch\mu_n + \cos\mu_n)\left[sh\left(\frac{\mu_n}{l}x\right) - \sin\left(\frac{\mu_n}{l}x\right)\right]}{sh\mu_n + \sin\mu_n}.$$

According to the general theory of the Sturm-Liouville problem functions $\{X_n(x)\}_{n=1}^{\infty}$ form a complete orthogonal system of functions on the interval $[0,l]$. Then the solution of problem (2) - (4) is following

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(a\frac{\mu_n^2}{l^2}t\right) + B_n \sin\left(a\frac{\mu_n^2}{l^2}t\right)\right] \cdot X_n(x),$$

where coefficients A_n And $B_n -$ are determined from the initial conditions by the formulas

$$A_n = \frac{\int_0^l f(x) X_n(x) dx}{\|X_n(x)\|^2}, \quad B_n = \frac{\int_0^l g(x) X_n(x) dx}{a\frac{\mu_n^2}{l}\|X_n(x)\|^2},$$

Where

$$\left\| X_n(x) \right\|^2 = \int_0^l X_n^2(x)\,dx.$$

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