SOLVING BOUNDARY VALUE PROBLEMS USING POTENTIAL THEORY. THEORETICAL FOUNDATIONS AND COMPUTATIONAL IMPLEMENTATIONS

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Boundary Value Problems (BVPs) are central to modeling physical phenomena such as electrostatic fields, steady-state heat conduction, fluid dynamics, and linear elasticity. While classical analytical methods like the separation of variables (Fourier method) are highly effective for simple geometries (e.g., spheres, cylinders, or rectangles), they fail when applied to domains with complex or arbitrary boundaries. To overcome this limitation, Potential Theory offers a powerful alternative. Instead of solving a differential equation inside a domain Ω, the problem is reformulated as an integral equation over the boundary $\partial\Omega$. This method possesses two extraordinary advantages.

Dimension Reduction: It reduces a d-dimensional spatial problem to a $(d-1)$-dimensional boundary problem, significantly lowering computational complexity.

Handling Unbounded Domains: It inherently satisfies radiation or decay conditions at infinity, making it exceptionally well-suited for exterior problems (e.g., acoustic or electromagnetic scattering).

Fundamental Solutions and Singularities. The cornerstone of potential theory is the concept of a *fundamental solution* (or free-space Green's function). For the Laplace operator Δ, the fundamental solution $U_0(x,y)$ represents the field generated by a point source at y in an infinite medium.

Mathematically, it satisfies the equation:

$$\Delta_x U_0(x,y) = -\delta(x-y)$$

where δ is the Dirac delta function. The explicit form of $U_0(x,y)$ depends on the spatial dimension n:

In 2D ($\mathbb{R}^2$):

$$U_0(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|}$$

In 3D ($\mathbb{R}^3$):

$$U_0(x,y) = \frac{1}{4\pi|x-y|}$$

Here, $r = |x-y|$ represents the Euclidean distance between the field point x and the source point y. The singularity at $x=y$ is the engine that allows integral representations to capture boundary behavior.

By convoluting the fundamental solution with various "density" distributions over the domain or its boundary, we define three classical potentials. For non-homogeneous equations like the Poisson equation ($\Delta u = -f$), the field generated by a continuous distribution of sources inside the domain Ω is given by the volume potential

$$V_\Omega(x) = \int_\Omega f(y) U_0(x,y)\, d\Omega_y$$

Single-Layer Potential. When a source density $\rho(y)$ is distributed continuously over a bounding surface $S = \partial\Omega$, it generates a single-layer potential:

$$V_S(x) = \int_S \rho(y) U_0(x, y) dS_y$$

Properties: $V_S(x)$ is continuous everywhere in $\mathbb{R}^n$, including when crossing the boundary S. However, its normal derivative experiences a sharp discontinuity (jump) across the surface.

Double-Layer Potential. If the boundary is populated by oriented dipoles with a moment density $\nu(y)$, the resulting field is a double-layer potential

$$W_S(x) = \int_S \nu(y) \frac{\partial U_0(x, y)}{\partial n_y} dS_y$$

where n_y is the outward unit normal vector to S at point y.

Properties: Unlike the single-layer potential, the double-layer potential $W_S(x)$ itself undergoes a finite jump when the point x crosses the boundary S.

Boundary Jump Relations and Fredholm Integral Equations. To solve BVPs, we must analyze the limits of these potentials as the interior or exterior field point approaches the boundary S. Let $x_0 \in S$, and let the subscripts $(+)$ and $(-)$ denote the limits from the interior (Ω_i) and exterior (Ω_e) domains, respectively.

The famous *Plemelj-Priwaloff jump relations* state that

$$W_S(x_0)_\pm = \pm\frac{1}{2}\nu(x_0) + \int_S \nu(y) \frac{\partial U_0(x_0, y)}{\partial n_y} dS_y$$

$$\left(\frac{\partial V_S(x_0)}{\partial n_{x_0}}\right)_\pm = \mp\frac{1}{2}\rho(x_0) + \int_S \rho(y) \frac{\partial U_0(x_0, y)}{\partial n_{x_0}} dS_y$$

Reduction of the Dirichlet Problem. Consider the *Interior Dirichlet Problem:* Find a function u such that $\Delta u = 0$ in Ω_i and $u|_S = g(x)$.

We seek the solution in the form of a double-layer potential, $u(x) = W_S(x)$. Applying the interior jump relation yields

$$\frac{1}{2}\nu(x_0) + \int_S \nu(y) \frac{\partial U_0(x_0, y)}{\partial n_y} dS_y = g(x_0), \quad x_0 \in S$$

This is a *Fredholm Integral Equation of the Second Kind* for the unknown density $\nu(y)$. According to Fredholm Alternative theorems, because the kernel is weakly singular, this equation possesses a unique, stable solution for any continuous boundary condition $g(x)$.

Modern Applications. The Boundary Element Method (BEM). While potential theory was historically used as an analytical tool to prove the existence and uniqueness of solutions, the advent of digital computing transformed it into a powerful numerical methodology known as the Boundary Element Method (BEM). In BEM, the boundary S is discretized into a finite number of elements (lines in 2D, triangles/quadrilaterals in 3D). The unknown densities (ν or ρ) are approximated using piecewise polynomial basis functions. The integral equation is then converted into a dense system of linear algebraic equations

$\mathbf{Ax} = \mathbf{b}$.

Potential theory bridges the gap between pure mathematical analysis and practical engineering simulation. By converting differential operators within a volume into integral operators over a surface, it offers an elegant approach to tackling complex boundaries and infinite domains.

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