PEDLOSKY BAROKLIN TO'LQIN AMPLITUDASI TENGLAMALARI

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Abstrakt. Ushbu maqolada ikki qatlamli geofizik suyuqlikdagi baroklin beqaror atrofidagi to'lqin paketlarining nochiziqli dinamikasi Pedlosky tomonidan taklif qilingan amplituda tenglamalari yordamida matematik tahlil qilingan. Gibbon, James va Morozning o'zgaruvchi almashtirish usuli yordamida Pedlosky tenglamalarining sine-Gordon tenglamasiga ekvivalentligi isbot qilinadi. AB tizimining matematik tuzilishi, saqlanish qonunlari, Darboux transformatsiyasi va soliton yechimlari etilgan.

Kalit so'zlar: *baroklin beqarorlik, AB tizimi, sine-Gordon tenglamasi, soliton, breather, halokatli to'lqin, Darboux transformatsiyasi.*

Geofizik suyuqliklarda baroklin beqarorlik hodisasi – doimiy zichlik sirtlarining gravitatsion potensial sirtlari bilan parallel bo'lmasligi natijasida yuzaga keladigan energiya almashinuvi jarayoni - atmosfera va okean dinamikasining markaziy muammolaridan biri hisoblanadi. Ushbu hodisa o'rta kengliklardagi g'arbiy shamollar, siklonlar, frontal tuzilmalar hamda okean girdoblari rivojlanishida hal qiluvchi rol o'ynaydi.

Pedlosky (1972) baroklin chegaraviy beqarorlik atrofidagi to'lqin paketlarining sekin o'zgaruvchi amplitudasini tavsiflovchi nochiziqli amplituda tenglamalarini hosil qildi. Gibbon, James va Moroz (1979) muhim kashfiyot qildi: Pedlosky tenglamalari tegishli o'zgaruvchi almashtirish orqali sine-Gordon tenglamasiga yoki atomik muhitlarda impulslarning tarqalishini tavsiflovchi tenglamalariga aylantirilishi mumkin ekan.

Gibbon va McGuinness (1981) tomonidan hosil qilingan AB tizimi – ikki qatlamli dissipativ suyuqlikning matematik modeli – nochiziqli evolyutsion tenglamalar nazariyasida muhim o'rin egallaydi. So'nggi yillarda ushbu tenglamalar tizimining turli xil aniq yechimlari, xususan breatherlar, halokatli to'lqinlar, solitonlar va ularning kombinatsiyalari o'rganilmoqda. Biroq, yuqori tartibli gibrid halokatli to'lqin va breather yechimlari hamda ular o'rtasidagi o'zaro ta'sirning nozik xossalari xususan, yarim elastik to'qnashuvlar hodisasi hali yetarlicha o'rganilmagan.

Pedlosky (1972) ikki qatlamli tizim uchun baroklin chegaraviy beqarorlik atrofidagi to'lqin paketlarining sekin o'zgaruvchi amplitudasini tavsiflovchi nochiziqli amplituda tenglamalarini keltirib chiqardi. Baroklin to'lqin quyidagi ko'rinishda ifodalanadi:

$$a(x,t) = A(X,T)e^{i(kx - \omega t)}, \tag{1}$$

bu yerda x, t – "tez" o'zgaruvchilar, $X = \varepsilon x$, $T = \varepsilon t$ – sekin o'zgaruvchilar, $\varepsilon \ll 1$ – kichik amplituda. $\omega = \omega(k)$ dispersiya munosabati klassik barotrop Rossby to'lqini uchun:

$$\omega = -\frac{\beta k}{k^2 + F}$$

bu yerda $\beta = \frac{\partial f}{\partial k}$ - beta-parametr, $F = f_0^2 L^2 / (g'H)$ - Froude soni (baroklin ta'sirni o'lchaydi), f_0 - Koriolis parametri, H - qatlam qalinligi.

Pedlosky amplituda tenglamalari quyidagi ko'rinishga ega:

$$\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)A=\sigma^2A-NAB, \tag{2}$$

$$\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)B=\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)A^2, \tag{3}$$

chegaraviy shartlar $A\to 0$ agar $X\to\infty$. Bu yerda:

$A(X,T)$ – kompleks baroklin to‘lqin amplitudasi;

$B(X,T)$ – haqiqiy funksiya, baroklin to‘lqin tomonidan asosiy oqimning o‘zgarish darajasini ifodalaydi;

c_1, c_2 – ikki to‘lqin rejimining guruh tezliklari;

$N>0$ – baroklin beqarorlik uchun zarur minimal kesimga bog’liq musbat son;

σ^2 – chiziqli o‘sish tezligi: $\sigma^2>0$ superkritik (beqaror), $\sigma^2<0$ subkritik (barqaror) holatga mos.

Tenglamada $B=0$ bo‘lganda quyidagi ko‘rinish oladi:

$$\left(\frac{\partial}{\partial T}+c_1\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+c_2\frac{\partial}{\partial X}\right)A=\sigma^2A.$$

Bunda $A\sim e^{i(KX-\Omega T)}$ shaklidagi tekis to‘lqinni qo‘yib, dispersiya munosabatini topamiz:

$$(\Omega-c_1K)(\Omega-c_2K)=-\sigma^2.$$

Bu kvadrat tenglama bo‘lib, uning yechimi:

$$\Omega=\frac{(c_1+c_2)K}{2}\pm\sqrt{\frac{(c_1-c_2)^2K^2}{4}-\sigma^2}$$

Agar $\sigma^2>0$ va $(c_1-c_2)^2K^2/4<\sigma^2$ bo‘lsa, Ω kompleks son bo‘ladi va beqarorlik yuzaga keladi. Beqarorlik sodir bo‘lishi uchun to‘lqin soniga shart:

$$|K|<K_c=\frac{2\sigma}{|c_1-c_2|}.$$

Gibbon, James va Moroz (1979) ning muhim kashfiyoti shundan iboratki, (2)-(3) Pedlosky tenglamalari tegishli o‘zgaruvchi almashtirish orqali *sine-Gordon* tenglamasiga aylantirilishi mumkin. Quyidagi yangi funksiyalar va koordinatalar kiritiladi:

$$S=\pm 1-\frac{NB}{\sigma^2}, \tag{5}$$

$$R=\sqrt{2}A, \tag{5}$$

$$\xi=-\frac{N^{1/2}(X-c_1T)}{c_1-c_2}, \tag{6}$$

$$\tau=\frac{\sigma^2N^{-1/2}(X-c_2T)}{c_1-c_2} \tag{7}$$

bu yerda belgilar $\sigma^2<0$ ga mos ravishda tanlanadi.

Ushbu almashtirishni qo'llash natijasida (2)–(3) tenglamalar quyidagi sodda ko'rinishga keladi:

$$R_{\xi\tau} = RS, \tag{8}$$

$$S_{\xi} = -\frac{1}{2}\left(R^2\right)_{\tau}. \tag{9}$$

Bu tenglamalar uchun muhim saqlanish qonuni mavjud:

$$S^2 + \left|R_{\tau}^{\,2}\right| = 1. \tag{10}$$

S funksiyasi tizimning mavjud potensial energiyasining o'lchovi sifatida talqin qilinadi. Superkritik holatda $S = +1$ - sof zonal oqim (maksimal APE), $S = -1$ - to'liq rivojlangan baroklin to'lqinlar (minimal APE). Subkritik holatda esa vaziyat teskari bo'ladi.

(9) ni ξ bo'yicha differensiallash va (8) ni ishlatish orqali:

$$S_{\xi\xi} = -\frac{1}{2}\left(|R|^2\right)_{\xi\tau} = -\frac{1}{2}\left(R^* R_{\xi\tau} + R^*_{\xi\tau} R\right)_{\tau} = -\left(\mathrm{Re}\left(R^* RS\right)\right)_{\tau} = -\left(S\,|R|^2\right)_{\tau}$$

Shuningdek, (10) ni ξ bo'yicha differensiallab

$$2SS_{\xi} + 2\mathrm{Re}\left(R_{\tau}^* R_{\xi\tau}\right) = 0$$

ga ega bo'lamiz. Bunga (8) va (9) ifodalarni qo'yib (10) bu munosabatning o'rinli ekanini ko'rish mumkin.

Agar R haqiqiy bo'lgan maxsus holni qarasak, $R = \varphi_{\xi}$, $S = \pm\cos\varphi$ almashtirishini qo'llash orqali:

$$\varphi_{\xi\tau} = \pm\sin\varphi. \tag{11}$$

Bu xarakteristik koordinatalardagi klassik sine-Gordon tenglamasi. Standart (x,t) koordinatalariga qaytish uchun $\xi = (x-t)/2, \tau = (x+t)/2$ almashtirishini qo'llash kifoya. Shu tarzda baroklin to'lqinlar masalasi soliton nazariyasidagi eng muhim tenglamalardan biriga keltirildi.

sine-Gordon tenglamasining bir soliton yechimi:

$$\varphi(\xi,\tau) = 4\arctan\left[\exp\left(\pm\frac{\xi/a + a\tau}{\sqrt{1-v^2}}\right)\right]$$

bu yerda v - soliton tezligi, a - masshtab parametri.

$$\varphi_{\text{kink}} = 4\arctan\left(e^{m(\xi - v\tau)}\right), \quad \varphi_{\text{antikink}} = 4\arctan\left(e^{-m(\xi - v\tau)}\right)$$

bu yerda $m = 1/\sqrt{1-v^2}$ - Lorentz omili.

$$\varphi_{\text{breather}} = 4\arctan\left[\frac{\sqrt{1-\omega^2}}{\omega}\cdot\frac{\sin\left(\omega\tau/\sqrt{1-\omega^2}\right)}{\cosh\left(\xi\sqrt{1-\omega^2}\right)}\right], \quad 0 < \omega < 1$$

Agar R kompleks bo'lsa, (2)–(3) tenglamalar sine-Gordon tenglamasining kompleks umumlashmasi bo'lib, atom muhitlarida ultra-qisqa optik impulslarning tarqalishini tavsiflovchi *SIT* (self-induced transparency) tenglamalari bilan aynan mosdir.

Gibbon va McGuinness (1981) ikki qatlamli dissipativ tizimni tavsiflash uchun quyidagi AB tizimini hosil qildi:

$$A_{xt} = \alpha A + \beta AB, \qquad (12)$$

$$B_x = -\frac{1}{2}\gamma |A^2|_t, \qquad (13)$$

bu yerda:

- x va t - normallashgan fazo va kechiktirilgan vaqt o'zgaruvchilari;
- $A(x,t) \in C$ - kompleks to'lqin qobig'i;
- $B(x,t) \in R$ - haqiqiy to'lqin qobig'i;
- $\alpha, \beta, \gamma \in R$ - tizimning doimiylari.

A va B funksiyalari quyidagi normallash shartini qanoatlantiradi:

$$|A_t^2| + |B^2| = 1.$$

AB tizimi (12)–(13) ikki qatlamli suyuqlik tizimidir va "AB tizimi" deb ataladi, chunki unda A va B harflari mos ravishda ikki xil qatlamdagi to'lqin harakatini ifodalaydi. Har bir qatlamdagi tezliklar doimiy bo'lsa-da, tezlik kesilishi (shear) nolga teng emas. Tizimning batafsil fizik asoslari manbada keltirilgan.

Agar $\alpha = 0$, $\beta = 1$, $\gamma = 1$ deb olinsa, AB tizimi (12)–(13) standart (soddalashtiriliangan) AB tizimiga aylanadi, bu holat adabiyotlarda kengroq o'rganilgan.

AB tizimi (12)-(13) integrallanuvchi bo'lib, uning Lax jufti quyidagicha:

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi \qquad (14)$$

bu yerda $\Psi = (\psi_1, \psi_2)^T$ - vektor funksiya, $\lambda \in \mathbb{C}$ - spektral parametr va

$$U = \begin{pmatrix} -i\lambda & \frac{i}{2}A \\ \frac{i}{2}A^* & i\lambda \end{pmatrix}, \quad V = \begin{pmatrix} \frac{i}{8\lambda}B & \frac{i}{4\lambda}A_t \\ \frac{i}{4\lambda}A_t^* & -\frac{i}{8\lambda}B \end{pmatrix}$$

(14) tenglamalarning birgalikdalik sharti $U_t - V_x + [U,V] = 0$ bo'lib u aynan AB tizimiga ekvivalentdir.

AB tizimi (12)-(13) quyidagi saqlanish qonunlariga ega:
(i) Birinchi integral:

$$\frac{\partial}{\partial x}(|A|^2) + 2\frac{\partial}{\partial t}(B) = 0$$

(ii) Normalizatsiya saqlanishi:

$$|A_t|^2 + B^2 = \text{const} = 1$$

(iii) Impuls:

$$I = \int_{-\infty}^{+\infty} |A|^2 dt = \text{const}$$

(iv) Energiya:

$$H = \int_{-\infty}^{+\infty} (|A_t|^2 - B|A|^2) dt = \text{const}$$

AB tizimi uchun Darboux transformatsiyasi quyidagicha quriladi. Faraz qilaylik, $\Psi^{(0)}$ - Lax juftining ma'lum asosiy yechimi bo'lib, $A^{(0)}, B^{(0)}$ - ma'lum (trivial yoki sodda) yechimlar bo'lsin. U holda yangi yechim quyidagi formula orqali olinadi:

$$A^{[1]} = A^{(0)} + \frac{2i(\lambda_1 - \lambda_1^*)\psi_1^{(1)}\psi_2^{(1)*}}{|\psi_1^{(1)}|^2 + |\psi_2^{(1)}|^2},$$

$$B^{[1]} = B^{(0)} + \frac{d}{dt}\left[\frac{(\lambda_1 - \lambda_1^*)\left(|\psi_1^{(1)}|^2 - |\psi_2^{(1)}|^2\right)}{|\psi_1^{(1)}|^2 + |\psi_2^{(1)}|^2}\right]$$

bu yerda $\psi_1^{(1)}, \psi_2^{(1)}$-spektral parametr $\lambda = \lambda_1$ bilan bog'liq Lax yordamchi funksiyasining komponentlari. n-marta Darboux transformatsiyasini qo'llash orqali determinant formulalar olinadi. n tartibli yechim:

$$A^{[n]} = A^{(0)} + 2i\frac{\det\left(\mathbf{M}_A^{[n]}\right)}{\det\left(\mathbf{M}^{[n]}\right)}$$

bu yerda $\mathbf{M}^{[n]}$ - $n \times n$ o'lchamli Darboux matritsasi:

$$M_{jk}^{[n]} = \frac{\psi_1^{(j)}\psi_1^{(k)*} + \psi_2^{(j)}\psi_2^{(k)*}}{\lambda_j - \lambda_k^*}, \quad j,k = 1,\cdots,n$$

Yassi to'lqin saklidagi $A^{(0)} = a_0 e^{i\theta}, B^{(0)} = -a_0^2/2$ asosiy yechimini qaraymiz, bu yerda $\theta = \kappa t + \mu x$ va dispersiya munosabati $\kappa\mu = \alpha - \beta a_0^2/2$. Lax yordamchi tenglamaning yechimi $\lambda_1 = \xi_1 + i\eta_1$ uchun:

$$\Psi^{(1)} = \begin{pmatrix} \cosh\Theta_1 + \frac{\delta_1}{\rho_1}\sinh\Theta_1 \\ \frac{ia_0 e^{i\theta}}{2\rho_1}\sinh\Theta_1 \end{pmatrix} e^{-i\xi_1 x + i\kappa_1 t}$$

bu yerda $\Theta_1 = \eta_1 x + \nu_1 t$, va ρ_1, δ_1, ν_1 - λ_1, a_0, κ orqali ifodalanadigan haqiqiy parametrlar. Natijada birinchi tartibli breather yechimi:

$$A^{[1]} = a_0 e^{i\theta}\left[1 + \frac{2i\eta_1\left(\cosh\Theta_1\sinh\Theta_1 + i\frac{\delta_1}{\rho_1}\sinh^2\Theta_1\right)}{\rho_1\left(\cosh^2\Theta_1 + \frac{\delta_1^2}{\rho_1^2}\sinh^2\Theta_1\right)}\right]$$

Bu yechim x yo'nalishida lokalizatsiyalangan va t bo'yicha davriy pulsatsiya qiluvchi to'lqinni ifodalaydi.

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