LIMIT THEOREMS FOR THE THEORY OF GALTON-WATSON BRANCHING SYSTEMS

Zuhriddin A. Nazarov
Munisa O. Ismoilova
Technology, Management and Communication Institute in Tashkent.
Department of "Languages, Exact and Social Sciences"

https://doi.org/10.5281/zenodo.20503697

Introduction. Stochastic branching systems are famous mathematical models describe a population size evolution of reproductive individuals. The Galton-Watson model, originally evolved as a family survival model in the second half of the 19th century, is the simple branching system. Modern branching systems models have arisen and progress due to modifications of the Galton-Watson model.

Let $Z(n)$ be a population size at the moment $n \in \mathbf{N}_0$ in the Galton-Watson branching (GWB) system with branching rates $\{p_k, k \in \mathbf{N}_0\}$, where $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$ and $\mathbf{N} = \{1, 2, \ldots\}$. Namely, each individual in the system lives a unit length life time and then gives $k \in \mathbf{N}_0$ descendants with probability p_k. This is a reducible, homogeneous-discrete-time Markov chain with a state space consisting of two classes: $S_0 = \{0\} \cup S$, where $S \subset \mathbf{N}$, therein the state $\{0\}$ is absorbing, and S is the class of possible essential communicating states.

We assume throughout this paper that $p_0 > 0$ and consider a Schröder case, i.e. $p_0 + p_1 > 0$. We also assume that

$$p_0 + p_1 < 1 \quad \text{and} \quad m := \sum_{k \in S} k p_k < \infty .$$

We are interested in the sub-critical and supercritical where $m < 1$ and $m > 1$ respectively. Denoting q be an extinction probability of the system starting from one individual, we recall that it is smallest root of the equation $f(s) = s$ on $s \in [0, 1]$, where

$$f(s) := \sum_{k \in S_0} p_k s^k$$

is the offspring generating function (GF). The extinction probability is 1 in sub-critical case and is less than 1 when the system is supercritical. So, the supercritical system survives with positive probability.

Put into consideration n-step transition probabilities

$$P_{ij}(n) := \mathrm{P}\{Z(n+k) = j \mid Z(k) = i\} \quad \text{for any} \quad k \in \mathbf{N}_0 .$$

A corresponding probability GF $\sum_{k \in S_0} P_{ij}(n) s^k = [f_n(s)]^i$, where

$$f_n(s) := \sum_{j \in S_0} p_j(n) s^j ,$$

therein $p_j(n) := P_{1j}(n)$. Needless to say that $f_n(0) = p_0(n)$ is a vanishing probability of the system starting from one individual. It tends as $n \to \infty$ monotonously to q, i.e. $\lim_{n\to\infty} p_0(n) = q$.

Consider a function $R_n(s) := q - f_n(s)$. It is clear that $R_n(0) \to 0$ as $n \to \infty$. Now denoting $H := \min\{n \in \mathbf{N} : Z(n) = 0\}$ be an extinction time of the system starting from single individual,

$$Q(n) := R_n(0) = P\{n < H < \infty\}$$

is a survival probability of the system in the finite time n. In sub-critical case we see $P\{H < \infty\} = 1$, hence $Q(n) = P\{Z(n) > 0\}$. In this case, A.Kolmogorov proved that if $f''(1-) < \infty$, then $Q(n)$ admits an asymptotic representation

$$Q(n) = K m^n (1 + o(1)) \quad as \quad n \to \infty, \qquad (1)$$

where K is the well-known Kolmogorov constant, but it does not have an explicit form. In this regard, Zolotarev expressed regret at the absence of an explicit expression for this constant. Formula (1) was later re-established under a weaker condition by A.Nagaev and I.Badalbaev, but and wherein the Kolmogorov constant remained not explicitly calculated. The Kolmogorov result (1) motivates to write an asymptotic representation $Q(n) = K_q b^n (1 + o(1))$ for the case $m \neq 1$, where

$$b := f'(q).$$

and K_q is a positive and it by right can be called the extended Kolmogorov constant.

Recently, the explicit form of K_q was calculated under the Kolmogorov conditions. Furthermore, formula (1) has been extended to $R_n(s)$ for the the case $m \neq 1$ and for all $s \in U_q[0, 1)$, where

$$U_q[0,1) := \{[0, q) \, U \, (q, 1)\}$$

is a unit interval with a punctured point q.

First, we recall the following Basic Lemma (but in a slightly different form) and essentially underlies the principal arguments in calculating the constant K_q.

Lemma 1. *Let* $m \neq 1$ *and* $f''(1-) < \infty$ *for* $m < 1$. *Then*

$$\sup\left\{ \left| \frac{b^n}{R_n(s)} - \frac{1}{A_q(s)} \right| : s \in U_q[0,1) \right\} = o(1), \qquad (2)$$

as $n \to \infty$, *where*

$$\frac{1}{A_q(s)} = \frac{1}{q-s} + \gamma_q ,$$

at that

$$\gamma_q := \frac{f''(q)}{2\beta(1-\beta)} .$$

Our principal result appears in next section. Here we improve the formulation of Lemma 1 by specifying the explicit approximation rate of the function $\beta^n / R_n(s)$ to the function $A_q(s)$.

Basic Lemma. Returning now to discussion in the proof of the Lemma 1 we recall that

$$M_0(n)\beta^n < \frac{\beta^n}{R_n(s)} - \frac{\beta^{n+1}}{R_{n+1}(s)} < M_1(n)\beta^n , \qquad (3)$$

where

$$M_k(n) := \frac{f''(q_{1-k}(n))}{2f'(q_k(n))}$$

and $q_k(n) := q + (k - q)\beta^n$ for $k = 0, 1$. Define

$$\beta_k := \frac{1}{k!} f^{(k)}(q) \quad \text{and} \quad B_k := \frac{\beta_k}{\beta}$$

for some $k \in \mathbf{N}$. It's obvious that $\beta_1 \equiv \beta$. In the following lemma we will establish an explicit expression of M assuming that $f'''(1-) < \infty$ for the sub-critical case.

Lemma 2. *Let* $f'''(1-) < \infty$ *for* $m < 1$. *Then the following asymptotic representation holds:*

$$\frac{\beta^n}{R_n(s)} - \frac{1}{q-s} = \gamma - B_2\beta^n + r_n(s) , \qquad (4)$$

where $\gamma = B_2/(1-\beta)$ *and* $r_n(s) = O\left(\beta^{2n}\right)$ *as* $n \to \infty$ *uniformly in* $s \in U_q[0,1)$.

Limit Theorems. The statements obtained in this section are the direct consequence from Lemma 2. Rewrite (4) as follows:

$$\frac{R_n(s)}{\beta^n} = A_q(s)\left[1 + B_2 A_q(s)\beta^n (1 + o(1))\right] \qquad (3.1)$$

as $n \to \infty$.

Theorem 1. *Let* $f'''(1-) < \infty$ *for* $m < 1$. *Then*

$$\frac{\pi_j(n)}{\pi_1(n)} = \pi_j \cdot (1 + r_n).$$

where $\gamma = B_2/(1-\beta)$ *and* $r_n(s) = O(\beta^{2n})$ *as* $n \to \infty$ *uniformly in* $s \in U_q[0,1)$.

Theorem 2. *Let* $f'''(1-) < \infty$ *for* $\mu < 1$. *Then*

$$\beta^{-n}\pi_j(n) = \pi_j \cdot (1 + r_n),$$

where $\gamma = B_2/(1-\beta)$ *and* $r_n(s) = O(\beta^{2n})$ *as* $n \to \infty$ *uniformly in* $s \in U_q[0,1)$.

Adabiyotlar, References, Литературы:

1.Johnson N.L., Kotz S., Kemp A.W. (2005). Univariate Discrete Distributions, Wiley.

2. Athreya K.B., Ney P.E. (2004). Branching Processes. Springer.

3. Cameron, A.C., Trivedi, P.K. (2013). Regression Analysis of Count Data. Cambridge University Press.

4. Kingman J.F.C. (1993). Poisson Processes. Oxford University Press.

5. Hilbe J.M. (2011). Negative Binomial Regression. Cambridge University Press.

6. McCullagh P., Nelder J.A. (1989). Generalized Linear Models. Chapman&Hall.

7. Ars Conjectandi by Jacques Bernoulli, published posthumously in 1713.

8. Borovkov A.A.: Probability Theory. URSS, Moscow, 2009.

9. Себастъянов Б.А.: Курс теории вероятностей и математическая статистика. Москва, Наука, 1982.

10. Феллер В.: Введение в теорию вероятностей и ее приложения. Москва, Мир, 1984.