

APPROXIMATE CALCULATION OF THE EXACT INTEGRAL

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ABSTRACT

The article provides detailed information about the concept of integral and the history of its origin. In addition to the Riemann integral, which is covered in the school program, a number of other integrals (Riemann-Stiltes, Daniell, Alfred Haar, Henstock-Kurzweil, Ito and Stratonovich, Chokvet) are described. The difference between integrals is explained with the help of concrete examples. The concept of definite integral and its application in practical problems are widely covered, counterexamples of some theorems are presented.

$\int_a^b f(x) dx$ in the above-mentioned methods of calculating the definite integral consisted in finding an initial function $F(x)$ of the function $f(x)$ and calculating its value. But for some definite integrals, we may encounter the following problems when applying these methods:

- 1) Finding the initial function $F(x)$ is complicated;
- 2) the initial function $F(x)$ is complex, and its $F(a)$ and $F(b)$ values are difficult to calculate;
- 3) the function $F(x)$ is not expressed in elementary functions;
- 4) The function $f(x)$ under the integral is given in table form.

In such cases, it is necessary to calculate the exact integral approximately. Various formulas have been found to solve this problem, and they are generally called quadrature formulas. Below are some of these formulas.

1. Formula of a rectangle. To derive this formula, we first divide the section $[a, b]$ into n equal parts with points $a = x_0 < x_1 < x_2 < \dots < x_n = b$. In this case, the length of each piece is equal to $\Delta x = \frac{b-a}{n}$ (Figure 1)

We denote the values of the function $f(x)$ under the integral at the points $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ by $y_0, y_1, y_2, \dots, y_{n-1}, y_n$ and the following we make sums:

$$y_0 \cdot \Delta x + y_1 \cdot \Delta x + y_2 \cdot \Delta x + \dots + y_{n-1} \Delta x; \quad y_0 \Delta x + y_1 \Delta x + y_2 \Delta x + \dots + y_n \Delta x. y_0 \cdot \Delta x + y_1 \cdot \Delta x + y_2 \cdot \Delta x + \dots + y_{n-1} \Delta x; \quad y_0 \Delta x + y_1 \Delta x + y_2 \Delta x + \dots + y_n \Delta x. y_0 \cdot \Delta x + y_1 \cdot \Delta x + y_2 \cdot \Delta x + \dots + y_{n-1} \Delta x; \quad y_0 \Delta x + y_1 \Delta x + y_2 \Delta x + \dots + y_n \Delta x.$$

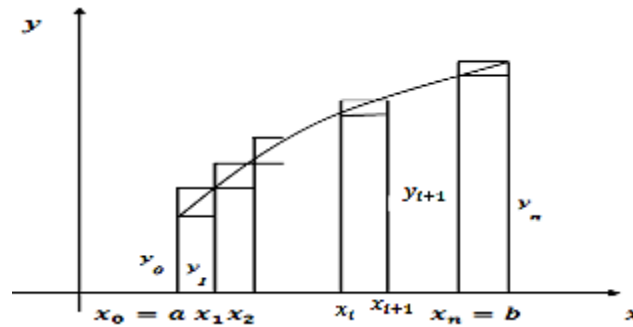


Figure 1

Each of these sums is an integral sum for the function $f(x)$ in the section $[a, b]$, and the following approximate formulas can be written for them:

$$\int_a^b f(x) dx \approx \frac{b-a}{n} (y_0 + y_1 + y_2 + \dots + y_{n-1})$$

$$\int_a^b f(x) dx \approx \frac{b-a}{n} (y_1 + y_2 + y_3 + \dots + y_n)$$

These formulas are called rectangular formulas.

II. Formula of trapezoids. In the formula for rectangles considered above, we replaced the curve $y=f(x)$ with stair lines. If we replace $y=f(x)$ with dashed lines inside, we get a more accurate value of the definite integral. In this case, the curved trapezoid $aABb$ consists of a set of trapezoids bounded from above by $AA_1, A_1A_2, \dots, A_{n-1}B$ vectors (Figure 2). The first in this

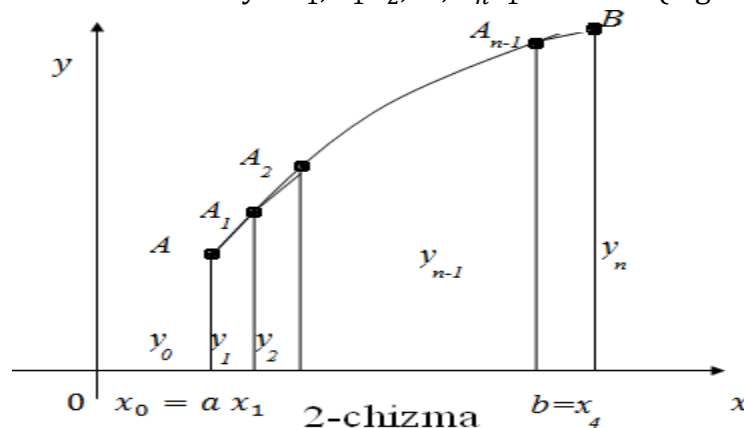


Figure 2

the face of a trapezoid $\frac{y_0+y_1}{2} \cdot \Delta x$, the other's face $\frac{y_1+y_2}{2} \cdot \Delta x$ and so on $\int_a^b f(x) dx \approx (\frac{y_0+y_1}{2} \cdot \Delta x + \frac{y_1+y_2}{2} \cdot \Delta x + \dots + \frac{y_{n-1}+y_n}{2} \cdot \Delta x)$ or

$$\int_a^b f(x) dx \approx \frac{b-a}{n} \left(\frac{y_0+y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right)$$

will be. This formula is called the trapezoidal formula for the approximation of the definite integral. Here the number n is chosen arbitrarily. The larger the number n , the more accurate the value of the integral.

III. Parabola formula (Simpson's formula). We divide the section $[a, b]$ into $n=2m$ equal parts. $[x_0, x_1]$ and $[x_1, x_2]$ are the faces of the curved trapezoids bounded by the curve $y=f(x)$

from the points $M_0(x_0, y_0)$, $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ we replace it with a curved trapezoid bounded by a passing parabola. Such a curved trapezoid is called a parabolic trapezoid (Figure 3).

The equation of a parabola whose axis is parallel to the Oy axis
It consists of $y = Ax^2 + Bx + C$.

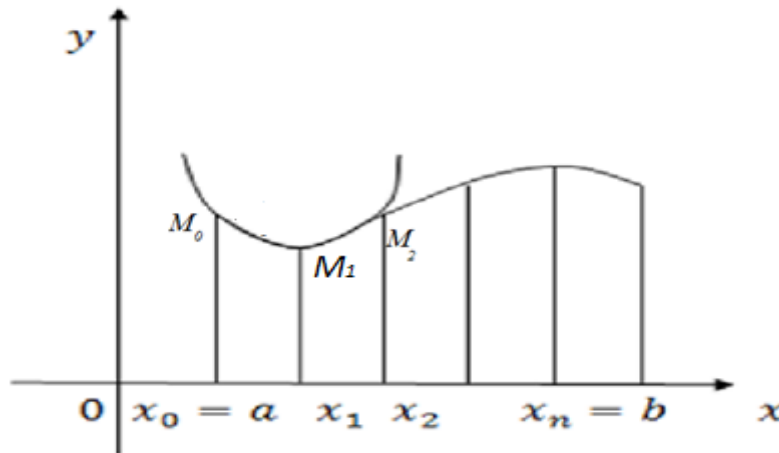


Figure 3

Coefficients A, B, C are found from the condition that the parabola passes through the given three points. We make parabolas as above for the rest of the sections. The sum of the faces of the formed parabolic trapezoids gives an approximate value of the integral. It consists of the following formula:

$$\int_a^b f(x) dx \approx \frac{b-a}{6m} [y_0 + y_{2m} + 2(y_2 + y_4 + \dots + y_{2m-2}) + 4(y_1 + y_3 + \dots + y_{2m-1})].$$

This formula is called Simpson's formula.

If the limit on the right side of the above equation exists and is finite, then the eigenintegral is called convergent, otherwise, it is called divergent.

In many cases, it is not necessary to know the exact value of the eigenintegral, it is enough to estimate whether it is converging or receding, and if it is converging, its value is sufficient.

I. If $a \leq x < \infty$ in the infinite semiinterval $0 \leq f(x) \leq g(x)$ and $\int_a^{+\infty} g(x) dx$ is an eigenintegral approximant, then $\int_a^{+\infty} f(x) dx$ The characteristic integral is also converging and the following inequality holds:

$$\int_a^{+\infty} f(x) dx \leq \int_a^{+\infty} g(x) dx$$

If the function $f(x)$ is defined in the interval $(-\infty; +\infty)$, its characteristic integral of type I over this interval is expressed as follows by the characteristic integrals introduced above.

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x) dx &= \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \\ &+ \lim_{b \rightarrow +\infty} \int_c^b f(x) dx. \end{aligned}$$



Now we generalize the concept of definite integral for unbounded functions. The given function $y=f(x)$ is not bounded on the half-interval $(a;b)$, but for arbitrary $\varepsilon \in (0, b-a]$, let this function be bounded and integrable on the section $[a+\varepsilon, b]$. In this case, $F(\varepsilon)F(\varepsilon) = \int_{a+\varepsilon}^b f(x) dx$, $\varepsilon \in (0, b-a]$ function can be viewed.

If the function $y=f(x)$ is not bounded at the point $x=a$, then $[a;+\infty)$ or $(-\infty;a]$ is determined by the following mixed-type proper integrals over infinite semiintervals:

$$\int_a^{+\infty} f(x) dx = \int_a^b f(x) dx + \int_b^{+\infty} f(x) dx \quad (a < b < +\infty)$$
$$\int_{-\infty}^a f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^a f(x) dx \quad (-\infty < c < a)$$

Examples:

1. $\int_0^1 \frac{dx}{1+x^2}$ calculate the integral using the rectangular formula.

Solving. For this, we divide the integration section $[0;1]$ into $n=10$ parts and present the results of calculations in the following table:

i	$x_i = 0.1i$	$1 + x_i^2$	$f(x_i) = \frac{1}{1 + x_i^2}$	$\sum_i f(x_i)$
1	0.1	1.01	0.9901	0.9901
2	0.2	1.04	0.9615	1.9516
3	0.3	1.09	0.9174	2.8690
4	0.4	1.16	0.8621	3.7311
5	0.5	1.25	0.8000	4.5311
6	0.6	1.36	0.7353	5.2664
7	0.7	1.49	0.6711	5.9375
8	0.8	1.64	0.6098	6.5473
9	0.9	1.81	0.5525	7.0998
10	1.0	2	0.5000	7.5998

Since $\Delta x = \frac{1-0}{10} = 0,1$ in our example, we get the following result based on the formula of a right rectangle.



$$\int_0^1 \frac{dx}{1+x^2} \approx 0,1 \cdot 7,5998 = 0,75998.$$

2. $\int_0^1 \frac{dx}{1+x^2}$ approximate the integral using the trapezoidal formula. Also take $n=10$.

Solving. $\frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10} = 0,1$; $\frac{y_0+y_n}{2} = \frac{1+0.5}{2} = 0,75$ va $y_1 + y_2 + \dots + y_9 = 7,0998$

$y_1 + y_2 + \dots + y_9 = 7,0998$ because

$$\int_0^1 \frac{dx}{1+x^2} \approx 0,1 \cdot [0,75 + 7,0998] = 0,78498.$$

3. $\int_1^9 \sqrt{6x-5} dx$ calculate the integral using Simpson's formula. In this case, take $n=8$.

Solving. $h = \frac{b-a}{n} = \frac{9-1}{8} = 1$. calculate the integral using Simpson's formula. In this case,

take $n=8$.

$$x_0 = 1, y_0 = \sqrt{6 \cdot 1 - 5} = \sqrt{1} = 1,0000,$$

$$x_1 = 2, y_1 = \sqrt{6 \cdot 2 - 5} = \sqrt{7} \approx 2.6458,$$

$$x_2 = 3, y_2 = \sqrt{6 \cdot 3 - 5} = \sqrt{13} \approx 3.6056,$$

$$x_3 = 4, y_3 = \sqrt{6 \cdot 4 - 5} = \sqrt{19} \approx 4.3589,$$

$$x_4 = 5, y_4 = \sqrt{6 \cdot 5 - 5} = \sqrt{25} = 5.0000,$$

$$x_5 = 6, y_5 = \sqrt{6 \cdot 6 - 5} = \sqrt{31} \approx 5.5678,$$

$$x_6 = 7, y_6 = \sqrt{6 \cdot 7 - 5} = \sqrt{37} \approx 6.0828,$$

$$x_7 = 8, y_7 = \sqrt{6 \cdot 8 - 5} = \sqrt{43} \approx 6.5574,$$

$$x_8 = 9, y_8 = \sqrt{6 \cdot 9 - 5} = \sqrt{49} = 7.0000.$$

These values found

$$\int_a^b f(x) dx \approx \frac{b-a}{6m} [y_0 + y_{2m} + 2(y_2 + y_4 + \dots + y_{2m-2}) + 4(y_1 + y_3 + \dots + y_{2m-1})]$$

These values found

We have $2m = 8$, $b - a = 9 - 1 = 8$ because $\frac{b-a}{6m} = \frac{8}{24} = \frac{1}{3}$. So, $\int_1^9 \sqrt{6x-5} dx \approx$
 $\frac{1}{3} [1 + 7 + 2(3.6056 + 5.0000 + 6.0828) + 4(2.6458 + 4.3589 + 5.5678 + 6.5574)] =$
 $\frac{1}{3} (8 + 2 \cdot 14.6884 + 4 \cdot 19.1299) = 37.9655.$ 4. Calculate the characteristic
 integral $\int_0^{+\infty} e^{-x} dx$.

Solving.

$$\int_0^{+\infty} e^{-x} dx = \lim_{b \rightarrow +\infty} \int_0^b e^{-x} dx = - \lim_{b \rightarrow +\infty} (-e^{-x}) \Big|_0^b = - \lim_{b \rightarrow +\infty} (e^{-x}) \Big|_0^b$$

$$= - \lim_{b \rightarrow +\infty} e^{-b} + \lim_{b \rightarrow +\infty} e^0 = - \lim_{b \rightarrow +\infty} \frac{1}{e^b} + 1 = 1.$$

$$\int_0^{+\infty} e^{-x} dx = \lim_{b \rightarrow +\infty} \int_0^b e^{-x} dx = - \lim_{b \rightarrow +\infty} (-e^{-x}) \Big|_0^b = - \lim_{b \rightarrow +\infty} (e^{-x}) \Big|_0^b =$$

$$- \lim_{b \rightarrow +\infty} e^{-b} + \lim_{b \rightarrow +\infty} e^0 = - \lim_{b \rightarrow +\infty} \frac{1}{e^b} + 1 = 1. \int_0^{+\infty} e^{-x} dx = \lim_{b \rightarrow +\infty} \int_0^b e^{-x} dx =$$

$$- \lim_{b \rightarrow +\infty} (-e^{-x}) \Big|_0^b = - \lim_{b \rightarrow +\infty} (e^{-x}) \Big|_0^b = - \lim_{b \rightarrow +\infty} e^{-b} + \lim_{b \rightarrow +\infty} e^0 = - \lim_{b \rightarrow +\infty} \frac{1}{e^b} + 1 = 1. \text{ So,}$$

the given property is an integral approximant.

5. Calculate the characteristic integral $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.



Solving. $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2} +$
 $+ \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{a \rightarrow -\infty} \arctg x \Big|_a^0 + \lim_{b \rightarrow +\infty} \arctg x \Big|_0^b =$
 $= \lim_{a \rightarrow -\infty} (\arctg 0 - \arctg a) + \lim_{b \rightarrow +\infty} (\arctg b - \arctg 0) =$
 $= -\lim_{a \rightarrow -\infty} \arctg a + \lim_{b \rightarrow +\infty} \arctg b = -\arctg(-\infty) + \arctg(+\infty) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \frac{\pi}{2} +$
 $\frac{\pi}{2} = \pi.$

Hence, the given eigenvalue is an integral approximant.

6. Calculate the characteristic integral $\int_0^1 \frac{dx}{x}$

Solving. At $x = 0$ the function under the integral has an infinite discontinuity. So, according to the definition

$$\int_0^1 \frac{dx}{x} = \lim_{a \rightarrow +0} \int_a^1 \frac{dx}{x} = \lim_{a \rightarrow +0} \ln x \Big|_a^1 + \lim_{a \rightarrow +0} (\ln 1 - \ln a) =$$
$$= -\lim_{a \rightarrow +0} \ln a = -(-\infty) = \infty.$$

Hence, the given integral is regressive.

7. Calculate the characteristic integral $\int_{-1}^2 \frac{dx}{\sqrt[3]{(x-1)^2}}$.

Solving. Here the function under the integral $[-1;2]$ has an infinite discontinuity at the internal point $x=1$ of the integration section. Therefore, by definition,

$$\int_{-1}^2 \frac{dx}{\sqrt[3]{(x-1)^2}} = \int_{-1}^1 \frac{dx}{\sqrt[3]{(x-1)^2}} + \int_1^2 \frac{dx}{\sqrt[3]{(x-1)^2}} = \lim_{a \rightarrow 1} \int_{-1}^a \frac{dx}{\sqrt[3]{(x-1)^2}} +$$
$$+ \lim_{a \rightarrow 1} \int_a^2 \frac{dx}{\sqrt[3]{(x-1)^2}} = \lim_{a \rightarrow 1} \int_{-1}^a (x-1)^{-\frac{2}{3}} d(x-1) +$$
$$+ \lim_{a \rightarrow 1} \int_a^2 (x-1)^{-\frac{2}{3}} d(x-1) = 3 \lim_{a \rightarrow 1} \sqrt[3]{x-1} \Big|_{-1}^a + 3 \lim_{a \rightarrow 1} \sqrt[3]{x-1} \Big|_a^2 =$$
$$= 3 \lim_{a \rightarrow 1} (\sqrt[3]{a-1} + \sqrt[3]{2}) + 3 \lim_{a \rightarrow 1} (\sqrt[3]{1} - \sqrt[3]{a-1}) = 3 \cdot \sqrt[3]{2} + 3 =$$
$$= 3(\sqrt[3]{2} + 1).$$

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