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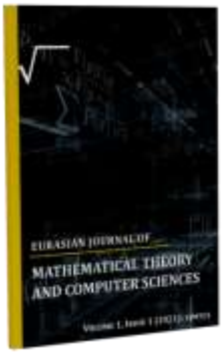

NUMERICAL SOLUTION OF THE EQUATION OF ION-ACOUSTIC WAVES IN A MAGNETIZED PLASMA

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ABSTRACT

Difference schemes for the equation of ion-acoustic waves in a magnetized plasma are proposed and investigated. The schemes are based on finite difference approximation in space and finite element approximation in time using fifth-degree polynomials. Theorems on the convergence of the constructed schemes are presented, as well as numerical calculations confirming the theoretical results obtained.

INTRODUCTION

Consider the following equation:

$$\frac{\partial^4}{\partial t^4}(\Delta u - r_D^{-2}u) + \frac{\partial^2}{\partial t^2}[(\omega_{B_i}^2 + \omega_{p_i}^2)\Delta u - \omega_{B_i}^2 r_D^{-2}u] + \omega_{p_i}^2 \omega_{B_i}^2 \Delta_3 = f(x,t), \qquad (1)$$

$$(x,t) \in \Omega = \{x = (x_1, x_2, x_3):\ 0 < x_k < l_k,\ k = 1,2,3\}.$$

The initial and boundary conditions have the following form:

$$\left.\frac{\partial^\nu}{\partial t^\nu} u(x,t)\right|_{t=0} = u_{0,\nu},\quad \nu = \overline{0,3},\quad x \in \bar{\Omega} = \Omega \cup \partial\Omega, \qquad (2)$$

$$u(x,t)\big|_{\partial\Omega} = 0,\ \ t \in (0,T]. \qquad (3)$$

Here $u = (x,t)$ is the speed of motion, $\Delta = \partial^2 u / \partial x_1^2 + \partial^2 u / \partial x_2^2 + \partial^2 u / \partial x_3^2$, $\Delta_3 = \partial^2 u / \partial x_3^2$, $r_D^2 = T_e^2 / (4\pi e^2 n_0)$ is the Debye radius, $\omega_{B_i} = eB_0/(Mc)$ is the ion gyro-frequency, $\omega_{p_i}^2 = 4\pi e^2 n_0 / M$ is the Langmuir frequency for ions, M is the mass, c is the speed of light in vacuum, B_0 is an external constant magnetic field, n_0 is the unperturbed particle density, e is the absolute value of the electron charge, T_e is the electron temperature.

Equation (1) refers to Sobolev-type equations [1], [2]. They appear when solving problems in geophysics, oceanology, atmospheric physics, the physics of magnetically ordered structures related to the wave propagation in media with strong dispersion, and in many other fields [1]–[3]. In addition, similar equations appear in the mathematical modeling of internal waves in the ocean and atmosphere [4]–[6]. The existence and uniqueness of solutions to such problems were studied in [7].

In our case, we will assume that $r_D^2 \notin \sigma(\Delta) = \lambda_k$ is the set of eigenvalues of the homogeneous Dirichlet problem for the Laplace operator in domain Ω.

The study in [3] is devoted to analytical methods for solving problems of this type, where the issues of global and local solvability of initial-boundary value problems for linear and nonlinear equations are considered. Besides, numerical methods for solving equations unresolved with respect to the time derivative were considered there. In [8], similar equations, with some replacements, were reduced to two equations (one contains differentiation in time, the other contains differentiation in space only), then these equations were solved by the finite difference method on quasi-uniform grids. At that, the second-order approximation for both variables is proved.

References [9], [10] are devoted to numerical methods for solving initial-boundary value problems for equation (1). In [8], a mathematical model of ion-acoustic waves in plasma in an external magnetic field was considered. Issues of unique solvability of the Cauchy-Dirichlet problem were studied. Based on the theoretical results, an algorithm was developed for the numerical solution of the problem based on the modified Galerkin method. An implementation algorithm was given. A study similar to an optimal control problem for the mathematical model (1) was conducted in [10], where an algorithm was developed for a numerical solution based on the modified Galerkin method and the Ritz method.

In this paper, we consider the issues of constructing and investigating difference schemes of higher accuracy of initial-boundary value problems for equations (1)-(3). The spatial variables are approximated by the grid method, and the temporal variable is stored in differential form. As a result, we obtain a system of ordinary high dimension differential equations, solved by the difference scheme of the finite element method of the fourth-order accuracy. To obtain an estimate of accuracy, a special technique for obtaining a priori estimates was used. Such studies for various stationary and nonstationary problems were conducted in [11]–[14]. The notation given in [15] is used in this paper.

DISCRETIZATION IN SPACE

We approximate equation (1) with respect to the spatial variables by the finite difference method. Let us introduce into $\bar{\Omega}$ the grid uniform in each direction $\bar{\omega}_h = \bar{\omega}_{h_1} \times \bar{\omega}_{h_2} \times \bar{\omega}_{h_3}$, where $\bar{\omega}_{h_\alpha} = \left\{ x_\alpha = i_\alpha h_\alpha, i_\alpha = \overline{0, N_\alpha},\ h_\alpha = l_\alpha / N_\alpha \right\}, \alpha = 1,2,3.$ Here $\bar{\omega}_h = \omega_h + \gamma_h$. Let us define subspace $H_h = \overset{\circ}{W}{}_2^1(\omega_h)$ - the space of grid functions $v(x_1, x_2, x_3)$ with norm $\|v\|_1^2 = \sqrt{\sum_{i_1}^{N_1}\sum_{i_2}^{N_2}\sum_{i_3}^{N_3} h_1 h_2 h_3 \left[(v_{\bar{x}_1})^2 + (v_{\bar{x}_2})^2 + (v_{\bar{x}_3})^2 \right]} \le M$, where constant M does not depend on $h_1,\ h_2,\ h_3$. Here $\overset{\circ}{W}{}_2^1(\omega_h)$ is the space of grid functions vanishing at the boundaries. Approximating the spatial variable on the specified grid, we pass from (1)-(3) to the definition of an approximate grid solution from the following Cauchy problem for function $u_h(t)$:

$$D\frac{d^4u_h(t)}{dt^4}+B\frac{d^2u_h(t)}{dt^2}+Au_h(t)=f_h(t),\quad \frac{d^\nu u_h}{dt^\nu}(0)=u_{0,\nu,h},\ \nu=\overline{0,3}. \qquad (4)$$

where

$$D=\Lambda-r_D^{-2}E,\ B=(\omega_{B_i}^2+\omega_{p_i}^2)\Lambda-\omega_{B_i}^2r_D^{-2}E,\ A=\omega_{p_i}^2\omega_{B_i}^2\Lambda_3, \qquad (5)$$

$\Lambda y=-\sum_{i=1}^{3}y_{x_i\bar{x}_i}$, $\Lambda_3 y=-y_{x_3\bar{x}_3}$, y is the function value at the fixed node x_α.

Difference operators $D,\ B$, and A approximate differential operators $\Delta-r_D^{-2}E$, $(\omega_{B_i}^2+\omega_{p_i}^2)\Delta-\omega_{B_i}^2r_D^{-2}E$ and $\omega_{p_i}^2\omega_{B_i}^2\Delta_3$ with the-second order of the approximation error. In addition, $D=D^*>0$, $B=B^*>0$, $A=A^*>0$. In what follows, for simplicity of notation, in (4) we use $u\in H_h$ instead of u_h, i.e., problem (4) is written in the following form

$$D\frac{d^4u}{dt^4}+B\frac{d^2u}{dt^2}+Au=f,\quad \frac{d^\nu u}{dt^\nu}(0)=u_{0,\nu},\ \nu=\overline{0,3}. \qquad (6)$$

The boundary conditions are approximated exactly.

DISCRETIZATION IN TIME

Let a discrete function y approximate a continuous function u. Here D, B, and A are determined according to (5). Let us discretize problem (6) by the finite element method relating values $y^{n+1},\ y^n,\ \dot{y}^{n+1},\ \dot{y}^n,\ \ddot{y}^{n+1},\ \ddot{y}^n$, approximating $u(t_n+\tau)$, $u(t_n)$, $\dot{u}(t_n+\tau)=du(t_n+\tau)/dt$, $\dot{u}_h(t_n)=du(t_n)/dt$, $\ddot{u}(t_n+\tau)=d^2u(t_n+\tau)/dt^2$, $\ddot{u}(t_n)=d^2u(t_n)/dt^2$, respectively. Such a scheme was constructed in [16] and it has the following form:

$$\begin{aligned} &D_\eta\dot{y}_t-\eta\tau^2Ay^{(0.5)}-D\ddot{y}^{(0.5)}=\varphi_1,\\ &D_\gamma y_t-D_\gamma\dot{y}^{(0.5)}+\eta\tau^2D\ddot{y}_t=\varphi_2,\\ &D_\alpha\dot{y}_t-D_\beta\ddot{y}^{(0.5)}-\eta\tau^2Ay^{(0.5)}=\varphi_3, \end{aligned} \qquad (7)$$

where $D_m=D-m\tau^2B$, $m=\alpha,\ \beta,\ \gamma,\ \eta$, $\varphi_1=-\frac{\tau}{6}\int_{t_n}^{t_{n+1}}f(t)dt=-\frac{\tau^2}{6}\int_0^1 f(t_n+\tau\xi)d\xi$,

$$\varphi_2=-\frac{7\tau}{60}\int_{t_n}^{t_{n+1}}f(t)\vartheta_2^{(\gamma,\eta)}(t)dt=-\frac{7\tau^2}{60}\int_0^1 f(t_n+\tau\xi)[s_1\vartheta_2^{(1)}(\xi)+s_2\vartheta_2^{(5)}(\xi)]d\xi,$$

$$\varphi_3=-\frac{10}{\tau}\int_{t_n}^{t_{n+1}}f(t)\vartheta_3^{(\alpha,\beta,\eta)}(t)dt=-10\int_0^1 f(t_n+\tau\xi)[s_3\vartheta_3^{(2)}+s_4\vartheta_3^{(4)}]d\xi,$$

$\vartheta_2^{(\gamma,\eta)}=s_1\vartheta_2^{(1)}+s_2\vartheta_2^{(5)}$, $\vartheta_2^{(1)}=\tau(\xi-1/2)$, $\vartheta_2^{(5)}=\tau(3\xi^5+15\xi^4/2-5\xi^3+\xi/2)$, $s_1=3-120\gamma$, $s_2=14-840\gamma$, $\vartheta_3^{(\alpha,\beta,\eta)}=s_3\vartheta_3^{(2)}+s_4\vartheta_3^{(4)}$, $\vartheta_3^{(2)}=\tau^2\xi(\xi-1)/2$, $\vartheta_3^{(4)}=\tau^2\xi^2(\xi-1)^2/4$, $s_3=140\alpha+15$, $s_4=1400\alpha+140$, here $\alpha,\ \beta,\ \gamma,\ \eta$ - are some constants.

The initial conditions are approximated as follows [15]:

$$y^0 = u_{0,0},\ \dot{y}^0 = u_{0,1} + \frac{\tau}{2}\left(E - \frac{\tau^2}{12}D^{-1}B\right)u_{0,2} + \frac{\tau^2}{6}u_{0,3} + \frac{\tau^3}{24}D^{-1}[f(0) - Au_{0,0}],$$

$$\ddot{y}^0 = u_{0,2} + \tau u_{0,3} + \frac{\tau^2}{2}D^{-1}[f(0) - Bu_{0,2} - Au_{0,0}] + \frac{\tau^3}{4}D^{-1}[\dot{f}(0) - B\dot{u}_{0,2} - A\dot{u}_{0,0}]. \quad (8)$$

It is easy to check that scheme (7), (8) has the fourth-order of approximation error on smooth solutions, i.e., $\psi_1 = O(\tau^4)$, $\psi_2 = O(\tau^4)$, $\psi_3 = O(\tau^4)$, if the following conditions are met:

$$\alpha - \beta = 1/12,\ \eta = 1/12, \quad (9)$$

and γ is the arbitrary constant.

STABILITY AND ACCURACY

Let us formulate a result on the stability and accuracy of the scheme. In [16], the accuracy estimate $\|u_h(t) - u(t)\|_A + \|\dot{u}_h(t) - \dot{u}(t)\|_D \le M\tau^4$ was obtained, provided that the stability condition of the scheme is met:

$$D - \mu\tau^2 A \ge \delta D,\quad 0 < \delta < 1,\ \mu = \max[\alpha, \beta, \gamma, \eta]. \quad (10)$$

By direct calculation, we can verify that (10) holds if the following condition is met:

$$\tau^2 \le (1 - \delta)/\mu. \quad (11)$$

Then, considering the mutual permutability of operators A, B, D [16], we establish the validity of the following assertion.

Theorem. Under conditions (9), (11) the solution of schemes (7), (8) converges to a sufficiently smooth solution of problem (1)-(3) and the following accuracy estimate holds

$$\|y(x,t) - u(x,t)\|_1 \le M(\tau^4 + h^2).$$

ALGORITHM FOR SCHEME IMPLEMENTATION

In [16], two methods for the numerical implementation of scheme (7), (8) were presented.

COMPUTATIONAL EXPERIMENT

Consider a one-dimensional case. Let the solution to problem (1)-(3) depend on one variable z only, i.e. $z = x_3$. Then equation (1) takes the following form

$$\frac{\partial^4}{\partial t^4}(\Delta u - r_D^{-2}u) + \frac{\partial^2}{\partial t^2}[(\omega_{B_i}^2 + \omega_{p_i}^2)\Delta u - \omega_{B_i}^2 r_D^{-2}u] + \omega_{p_i}^2\omega_{B_i}^2\Delta u = f(z,t), \quad (12)$$

$$u = u(z,t),\ \ (z,t) \in \Omega = \{z:\ 0 < z < l\}.$$

The initial and boundary conditions have the following form:

$$\left.\frac{\partial^\nu}{\partial t^\nu}u(z,t)\right|_{t=0} = u_{0,\nu},\quad \nu = \overline{0,3},\quad z \in \bar{\Omega} = \Omega \cup (0,l), \quad (13)$$

$$u(0,t) = u(l,t) = 0,\ t \in (0,T]. \quad (14)$$

It is required to find a numerical solution to problem (1)-(3) by scheme (7), (8) for the following values: $r_D = \omega_{B_i} = \omega_{p_i} = 1$, $l = \pi$, $u(0,t) = u(\pi,t) = 0$, $u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = \sin z$ on segment $[0,\pi]$. The exact solution to problem (12)-(14) has the following form $u(z,t) = e^t \sin z$. The right part is $f(z,t) = -e^t \sin z(5e^t \sin z + 1)$. The scheme parameters are chosen from the

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stability and accuracy of the scheme, for example, for $\alpha = 1/10$, $\beta = 1/60$, $\gamma = 1/40$, $\eta = 1/12$ we have $\mu = 1/10$. Then the stability condition for the scheme takes the following form $\tau \le \sqrt{10(1-\delta)}$.

Table 1. *Rates of convergence in spatial direction*

Space step	Time step	Error	Order of accuracy
h = 0.01	τ = 0.01	2.31E-08	
h = 0.005	τ = 0.01	5.68E-09	2.02393
h = 0.0025	τ = 0.01	1.42E-09	2
h = 0.00125	τ = 0.01	3.53E-10	2.008151

Table 2. *Rates of convergence in time direction*

Space step	Time step	Error	Order of accuracy
h = 0.01	τ = 0.01	2.31E-08	
h = 0.01	τ = 0.005	1.44E-09	4.006259
h = 0.01	τ = 0.0025	9.00E-11	3.997493
h = 0.01	τ = 0.00125	5.61E-12	4.004013

Let us determine the rates of convergence along the spatial and temporal directions by the following formulas $\log_2(z(2h,\tau)/z(h,\tau))$ and $\log_2(z(h,2\tau)/z(h,\tau))$. The results are shown in Tables 1 and 2.

CONCLUSIONS

A numerical method with a high degree of accuracy for solving an initial-boundary value problem for a mathematical model of the ion-acoustic wave propagation in a magnetized plasma was developed and investigated in this article. The stability and convergence of the constructed methods were proved, and accuracy estimates were obtained on their basis. The schemes were tested by computational experiment. The computational experiment conducted illustrated the efficiency of the difference scheme built on the basis of the finite element method of the fourth-order of accuracy in time. The results obtained can find further application in the study of other similar initial-boundary value problems, including local and non-local ones.

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