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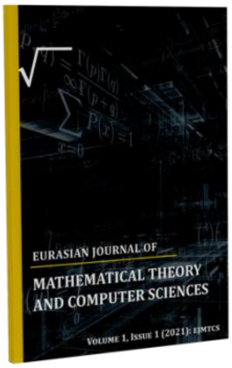

INITIAL-BOUNDARY PROBLEM FOR DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVES OF HIGHER ORDER IN THE SOBOLEV CLASS

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https://doi.org/10.5281/zenodo.16927356

ARTICLE INFO

Received: 16th August 2025
Accepted: 21st August 2025
Online: 22nd August 2025

KEYWORDS

Initial-boundary value problem, fractional derivative, initial-boundary value problem in a cylindrical domain, fractional partial differential equation, higher-order differential equations in the Sobolev class, fractional differential equations in the Sobolev class.

ABSTRACT

Due to the large applicability of fractional equations to physical, technical, and biological processes, scientists around the world are increasingly interested in the study of higher-order equations involving fractional derivatives. Today, the study and solution of initial boundary value problems for higher-order mixed-type equations involving fractional derivatives has become an urgent task. In this work, the initial-boundary value problem in a cylindrical domain for a partial differential equation involving a fractional derivative in the Miller-Ross sense and the initial-boundary value problem for a higher-order fractional differential equation in the Sobolev class are investigated.
Spherical functions are a method that allows us to find solutions to problems in mathematical physics more simply, easily, and quickly. With the help of these functions, it is possible to easily find solutions to even more complex problems.

SOBOLEV SINFIDA YUQORI TARTIBLI KASR HOSILALI DIFFERENSIAL TENGLAMALAR UCHUN BOSHLANG’ICH-CHEGARAVIY MASALA

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https://doi.org/10.5281/zenodo.16927356

ARTICLE INFO

Received: 16th August 2025
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ABSTRACT

Kasr tartibli tenglamalarning fizik, texnik va biologik jarayonlarga tadbiqi katta boʻlgani uchun butun dunyo olimlari tomonidan kasr tartibli hosila qatnashgan yuqori tartibli tenglamalarni oʻrganishga boʻlgan qiziqish ortib bormoqda. Bugungi kunda kasr tartibli hosila qatnashgan yuqori tartibli aralash tipdagi tenglamalar uchun boshlang’ich chegaraviy masalalarni oʻrganish va yechish dolzarb masalaga aylandi. Ushbu ishida Miller-Ross ma’nosidagi kasr tartibli hosila qatnashgan ayrim xususiy

KEYWORDS

Boshlang'ich-chegaraviy masala, kasr tartibli hosila, silindrik sohadagi boshlang'ich-chegaraviy masala, kasr tartibli xususiy hosilali differensial tenglama, Sobolev sinfida yuqori tartibli differensial tenglamalar, sobolev sinfida kasr tartibli differensial tenglamalar.

hosilali differensial tenglama uchun silindrik sohadagi boshlang'ich – chegaraviy masala va Sobolev sinfida yuqori tartibli kasr hosilali differensial tenglama uchun boshlang'ich – chegaraviy masalalar oʻrganilgan.

Sferik funksiyalar bizga matematik fizika masalalarining yechimlarini soddaroq, oson va tezroq topishga imkon yaratadigan usuldir. Bu funksiyalar orqali bir muncha murakkab boʻlgan masalalarni ham osongina yechimini topish mumkin.

Ushbu maqolada $Q = U \times (0, T)$ shohada, bunda U markazi koordinata boshida boʻlgan R radiusli uch oʻlchamli shar, T – esa, oldindan berilgan musbat son boʻlib, quyidagi

$$D_j^{\alpha} u(x,y,z,t) - a^2 \left(\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} \right) = 0,$$

$$(x,y,z,t) \in Q, \quad l-1 \le \alpha < l, \quad 0 \le j \le n-1, \quad l,\ n,\ j+1 \in \mathbb{N} \tag{1}$$

koʻrinishdagi tenglamani

$$\begin{cases} D_{j-i-1}^{\alpha-i-1} u(x,y,z,t)\Big|_{t=0} = \tilde{\varphi}_i^0(x,y,z),\ i = 0,\dots, j-1, \\ \left.\dfrac{\partial^s u(x,y,z,t)}{\partial x^s}\right|_{t=0} = \varphi_s^0(x,y,z),\ s = 0,\dots, l-j-1 \end{cases} \tag{2}$$

boshlang'ich shart va

$$u(x,y,z,t)\big|_{(x,y,z)\in \partial U} = 0 \tag{3}$$

chegaraviy shartlar bilan qaraymiz, bunda D_j^{α} Riman–Liuvill boʻyicha Miller–Ross ma'nosidagi hosila boʻlgan integro–differensial operatordir.

$f(t):\ \mathbb{R}_+ \to \mathbb{R}^m$ qandaydir l marta uzluksiz differensiallanuvchi funksiya boʻlsin. $l-1 \le \alpha < l,\, l \in \mathbb{N}$ uchun α – tartibli Riman–Liuvill boʻyicha kasr tartibli hosilani

$$D^{\alpha} f(t) = \frac{1}{\Gamma(l-\alpha)} \left(\frac{d}{dt} \right)^l \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-l+1}} d\tau \tag{4}$$

tenglik bilan kiritamiz, bunda $\Gamma(\cdot)$ – gamma-funksiya boʻlib, $\Gamma(z+1) = z\Gamma(z)$ tenglikni qanoatlantiradi.

Ushbu

$$D^{\alpha} f(t)=\sum_{k=0}^{l-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0)+\frac{1}{\Gamma(l-\alpha)} \int_0^t \frac{f^{(l)}(\tau)}{(t-\tau)^{\alpha-l+1}} d\tau \tag{5}$$

formula o‘rinlidir.

Shunday qilib, α – tartibli Riman–Liuvill bo‘yicha kasr tartibli hosilani

$$\sum_{k=0}^{l-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) \tag{6}$$

singulyar hadlar va

$$\frac{1}{\Gamma(l-\alpha)} \int_0^t \frac{f^{(l)}(\tau)}{(t-\tau)^{\alpha-l+1}} d\tau \tag{7}$$

integrallarning yig’indisi ko‘rinishida tasvirlash mumkin. Bu oxirgi (7) integral Kaputo ma’nosidagi α – kasr tartibli hosila deb, yo‘ki regulyarlashgan kasr tartibli hosila deb aytiladi. Ushbu (6) ko‘rinishdagi hadlarning qatnashishi α – tartibli Riman–Liuvill bo‘yicha kasr tartibli hosilaning nol nuqtada maxsuslikga egaligini bildiradi va α – tartibli Riman–Liuvill bo‘yicha kasr tartibli hosilali differensial tenglama uchun Koshi masalasini qo‘yishda maxsus ko‘rinishdagi bo‘shlang’ich shartlarni berish zarur bo‘ladi.

Riman–Liuvill bo‘yicha kasr tartibli hosilaning bunday kamchiligi $l-1 \le \alpha < l,\, l \in \mathbb{N}$ uchun α – tartibli regulyarlashgan kasr tartibli hosila

$$D^{(\alpha)} f(t)=\frac{1}{\Gamma(l-\alpha)} \int_0^t \frac{f^{(l)}(\tau)}{(t-\tau)^{\alpha-l+1}} d\tau = D^{\alpha} f(t)-\sum_{k=0}^{l-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) \tag{8}$$

ko‘rinishida bo‘lib, birinchi bor Kaputo ishlarida, hamda bo‘liq bo‘lmagan holda Djarbashyan va Nersesyan ishlarida paydo bo‘lgan.

Bu Riman–Liuvill bo‘yicha kasr tartibli hosila va Kaputo bo‘yicha kasr tartibli hosila ham yarim gruppa xossalarini, shuningdek kommutativlik xossalarini ham qanoatlantirmaydi, ya’ni

$$\mathbf{D}^{\alpha+\beta} f(t) \neq \mathbf{D}^{\alpha}\mathbf{D}^{\beta} f(t), \qquad \mathbf{D}^{\alpha}\mathbf{D}^{\beta} f(t) \neq \mathbf{D}^{\beta}\mathbf{D}^{\alpha} f(t)$$

bo‘ladi, bunda $\mathbf{D}^{\alpha}$ orqali Riman–Liuvill bo‘yicha kasr tartibli hosila, yoki Kaputo bo‘yicha kasr tartibli hosilani bildiradi. Shu sababdan, Miller va Ross tomonidan sekventsial hosila deb ataluvchi quyidagi ta’rif kiritilgan:

$$\mathrm{D}^{\alpha} f(t)=\mathbf{D}^{\alpha_1}\mathbf{D}^{\alpha_2}\ldots\mathbf{D}^{\alpha_m} f(t), \tag{9}$$

bunda $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_m)$ – mul’tiindeks, $f(t)$ funksiya esa, yeatarlicha uzluksiz differensiallanuvch deb olinadi. Umuman olganda, Miller–Ross sekventsial hosilasining asosida joylashgan $\mathbf{D}^{\alpha}$ operator sifatida Riman–Liuvill bo‘yicha kasr tartibli hosila, yoki Kaputo bo‘yicha kasr tartibli hosilani, yoki boshqa shu ko‘rinishdagi operatorlarni qo‘llash mumkin. Xususan, α_i butun son uchun $\left(\frac{d}{dt}\right)^{\alpha_i}$ oddiy differensiallash operatori bo‘lishi mumkin. Miller–

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Ross sekventsial hosilasining qo‘llanilishi, xususan differensial tenglamaning tartibini pasaytiradi.

Qandaydir $\nu,\ l-1\le\nu<l,\ l\in\mathbb{N}$ sonni tanlaymiz. $\alpha=(j,\nu-l+1,l-1-j)$, $j=0,\dots,\ l-1$ bo‘lgan holda to‘xtalamiz.

belgilashlarni kiritamiz, bunda $j=0,\dots,\ l-1$.

Shunday qilib, (1)–(3) boshlang’ich chegaraviy masalaning yechimi

$$u(r,\theta,\varphi,t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left[\sum_{s=0}^{l-j-1}t^{s}E_{\frac{1}{\alpha}}\left(-\lambda_{mn}\cdot t^{\alpha};s+1\right)\varphi_{s;m,n,j}^{0}+\right.$$
$$\left.+\sum_{i=0}^{j-1}t^{\alpha-i-1}E_{\frac{1}{\alpha}}\left(-\lambda_{mn}\cdot t^{\alpha};\alpha-i\right)\tilde{\varphi}_{i;m,n,j}^{0}\right]\frac{\psi_{n}(kr)}{\psi_{n}(kR)}Y_{n}^{(j)}(\theta,\varphi) \qquad (10)$$

ko‘rinishida hosil bo‘ladi, bunda

$$E_{\eta}(x;\mu)=\sum_{k=0}^{\infty}\frac{x^{k}}{\Gamma(k\eta^{-1}+\mu)}$$

Mittag –Leffler funksiyasidir.

Masalaning qo‘yilishi. Strukturaviy mexanikada katta ahamiyatga ega bo‘lgan sterjen, balka va plastinka tebranishlarining ko‘plab masalalari yuqori tartibli differensial tenglamalarga olib kelinadi. Balka tenglamasiga olib kelinuvchi vallar barqarorligini hisoblash va kema tebranishlari o‘rganilgan. Bu ishda $Q=\Pi\times(0,T)$ sohada $\Pi=(0,l)\times\ldots\times(0,l)$, $l,T-$ lar esa oldindan berilgan musbat sonlar,

$$D_{j}^{\alpha}u(y,t)+\alpha^{2}\sum_{p=1}^{N}\frac{\partial^{4m}u(y,t)}{\partial y_{p}^{4m}}=f(y,t),\ (y,t)\in Q,\ n-1\le\alpha<n,\ j=\overline{0,n-1},\ m,n\in N$$

(11)

tenglamani

$$\begin{cases}D_{j-i-1}^{\alpha-i-1}u(y,t)|_{t=0}=\varphi_{i}^{-0}(y),\ i=0,\dots,\ j-1,\\ \dfrac{\partial^{s}u(y,0)}{\partial y^{s}}=\varphi_{s}^{0}(y),\ s=0,\dots,n-j-1.\end{cases}, \qquad (12)$$

boshlang’ich shartlar va

$$\begin{cases}\dfrac{\partial^{4k}u(y,t)}{\partial^{s}y_{p}^{4k}}|_{y_{p}=0}=0,\ \dfrac{\partial^{4k+2}u(y,t)}{\partial^{s}y_{p}^{4k+1}}|_{y_{p}=0}=0,\\ \dfrac{\partial^{4k+2}u(y,t)}{\partial^{s}y_{p}^{4k}}|_{y_{p}=l}=0,\ \dfrac{\partial^{4k+3}u(y,t)}{\partial^{s}y_{p}^{4k+2}}|_{y_{p}=l}=0,\ \ k=\overline{0,m-1},\ \ p=\overline{1,N}\end{cases} \qquad (13)$$

chegaraviy shartlar bilan qaraymiz. Bu yerda, $(y,t)=(y_1,...,y_p,...,y_N,t)\in Q$, $a>0$ fiksirlangan, $f(y,t),\varphi_i^{\sim 0}(y),i=0,...,j-1,$ va $\varphi_s^0(y),\ s=0,...,n-j-1-$ esa yetarli darajada silliq funksiyalar va xos funksiyalar bo'yicha qatorga yoyish mumkin $v_{m_1,...,m_N}(x_1,...,x_N)=\prod_{p=1}^{N}X_{m_p}(x_p)$, bu yerda

$$X_{m_p}(x_p)=\frac{1}{\sqrt{1+b_{m_p}^{4s_p}}}\frac{1}{\sqrt{l}\left|\sin b_{m_p}\right|}\left(\sin b_{m_p}x_p+\sigma_{m_p}shb_{m_p}x_p\right),\ m_j\in\overline{Z}_+,$$

b_{m_p} -tenglamaning ildizi $tg(lb)=th(lb)$, $\sigma_{m_p}=\dfrac{\cos\left(b_{m_p}l\right)}{ch\left(b_{m_p}l\right)}$

. D_j^{α} Riman–Liuvill bo'yicha Miller–Ross ma'nosidagi sekvensial hosila bo'lgan integro–differensial operatordir.

N o'zgaruvchili $f(x)=f(x_1,...,x_N)$ funksiya $\overset{\circ}{W}{}_2^{2s_1,2s_2,2s_3,...,2s_N}(\varPi)$ fazoda barcha $\{v_n(y),n\in\overline{Z}_+^N\}$ lar bilan to'la ortonormal sistema tashkil qiladi. Shunday qilib, quyidagi teorema o'rinli.

Teorema. Aytaylik $\varphi_i^{\sim 0}(y),\ i=0,...,j-1,\varphi_i^0(y),\ s=0,...,n-j-1$ boshlang'ich shartlar va (11) tenglikning o'ng qismi $f(y,t)$

$$\sum_{m_1=0}^{\infty}...\sum_{m_N=0}^{\infty}|\sum_{s=0}^{n-j-1}t^sE_{\frac{1}{\alpha}}(\mu_{m_1,...,m_N}\cdot t^{\alpha};s+1)\varphi_{s;m_1,...,m_N}^{0}+\sum_{k=0}^{j-1}t^{\alpha-k-1}E_{\frac{1}{\alpha}}(\mu_{m_1,...,m_N}\cdot t^{\alpha};\alpha-k)\varphi_{s;m_1,...,m_N}^{\sim 0}+$$
$$+\int_0^l(t-\tau)^{\alpha-1}E_{\frac{1}{\alpha}}(\mu_{m_1,...,m_N}\cdot t^{\alpha};(t-\tau)^{\alpha-1})f_{m_1,...,m_N}(\tau)d\tau|^2\cdot\prod_{k=1}^{N}(1+b_{m_k}^{2s_k})<\infty$$

har bir $t>0$ uchun shartlarni qanoatlantirsin. U holda (11), (12), (13) masalaning regulyar yechimi $s_1=s_2=...=s_N>4m+\frac{N}{2},\theta=-[-\alpha]$ ko'rsatkichli $\overset{\circ}{W}{}_2^{s_1,s_2,s_3,...,s_N}(Q)$ fazoda mavjud, yagona va uni quyidagi ko'rinishda ifodalash mumkin

$$u(y,t)=\sum_{m_1=0}^{\infty}...\sum_{m_N=0}^{\infty}[\sum_{s=0}^{n-j-1}t^sE_{\frac{1}{\alpha}}(\mu_{m_1,...,m_N}\cdot t^{\alpha};s+1)\varphi_{s;m_1,...,m_N}^{0}+$$
$$+\sum_{k=0}^{j-1}t^{\alpha-k-1}E_{\frac{1}{\alpha}}(\mu_{m_1,...,m_N}\cdot t^{\alpha};\alpha-k)\varphi_{s;m_1,...,m_N}^{\sim 0}+$$
$$+\int_0^l(t-\tau)^{\alpha-1}E_{\frac{1}{\alpha}}(\mu_{m_1,...,m_N}\cdot t^{\alpha};\alpha)f_{m_1,...,m_N(\tau)d\tau}]\cdot\tilde{v}_{m_1,...,m_N(y_1,...,y_N)}$$

bundagi

$$\mu_{m_1,\dots,m_N} = -\lambda_{m_1,\dots,m_N} = -a^2\sum_{j=1}^{N}\lambda_{m_j} = -a^2\sum_{j=1}^{N}b_{m_j}^{4m},\ E_{\frac{1}{\alpha}}(\mu_{m_1,\dots,m_N}\cdot t^{\alpha}l\alpha-k)=\sum_{q=0}^{\infty}\frac{(\mu_{m_1,\dots,m_N}\cdot t^{\alpha})^q}{\Gamma(\alpha q+\alpha-k)}$$

koeffisientlarni formulalar bilan aniqlanadi.

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