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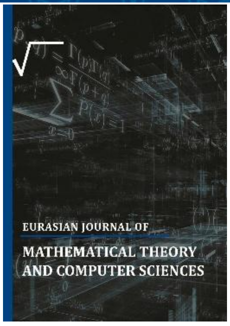

ON DERIVATIONS, 2-LOCAL OPERATORS, AND CENTRAL EXTENSIONS OF JORDAN-BANACH ALGEBRAS

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ABSTRACT

This article investigates structural properties of Jordan-Banach algebras, with emphasis on continuous derivations, 2-local derivations, and cohomological invariants. We establish that every 2-local derivation on a semi-simple Jordan-Banach algebra with essential socle is a continuous derivation, extending classical results of Šemrl and Ayupov to the non-associative topological setting. Furthermore, we prove that the second continuous cohomology group $H^2(A,A)$ *governs formal rigidity of Jordan-Banach algebras under infinitesimal deformations, and demonstrate that the vanishing of* H^2 *provides an obstruction-free criterion for triviality of central extensions. As applications, we characterize derivations on Jordan spin factors and modular annihilator Jordan-Banach algebras. The results bridge algebraic classification techniques with functional analytic frameworks, offering tools for deformation theory in algebraic quantum mechanics.*

Introduction

The study of Jordan algebras originated in the foundational work of Jordan, von Neumann, and Wigner [1] in their axiomatization of quantum mechanical observables. Unlike associative algebras, Jordan algebras satisfy the commutative identity $x^\circ y = y^\circ x$ and the Jordan identity $(x^\circ y)^\circ x^2 = x^\circ(y^\circ x^2)$, rendering them a fundamental non-associative structure in modern algebra and mathematical physics.

When equipped with a complete norm compatible with the algebraic operations, a Jordan algebra becomes a Jordan-Banach algebra [5]. The investigation of derivations—linear maps satisfying the Leibniz rule—and central extensions of such algebras has attracted sustained attention due to their roles in deformation theory and representation theory.

The concept of 2-local derivations was introduced by Šemrl [18] for algebras of bounded operators on Hilbert space: a (not necessarily linear) mapping

$\Delta : A \to A$ is a 2-local derivation if for every $x, y \in A$, there exists a derivation $D_{x,y} : A \to A$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. This notion has been extensively studied for associative Banach algebras [1][2][3][4] and, more recently, for finite-dimensional Jordan algebras [7][8]. However, the infinite-dimensional Jordan-Banach setting remains largely unexplored.

The main novelty of this paper is twofold:

- First, we establish that every 2-local derivation on a semi-simple Jordan-Banach algebra A with essential socle is automatically a continuous global derivation (Theorem 4.1). This extends the results of Ayupov, Arzikulov, and collaborators [7][8] from finite-dimensional Jordan algebras to the infinite-dimensional Banach setting, and generalises the associative Banach algebra results of [1] to the Jordan category.
- Second, we prove that the vanishing of the second continuous cohomology group $H^2(A, A)$ is equivalent to formal rigidity of the Jordan-Banach algebra structure under continuous deformations (Theorem 6.2). This establishes a cohomological criterion for rigidity in the Jordan-Banach framework, complementing the classical results of Gerstenhaber [9] for associative algebras.

These results provide a comprehensive structural and cohomological framework for Jordan-Banach algebras, with applications to algebraic quantum mechanics and deformation theory.

Preliminaries. Throughout, let $\mathbb{K}$ denote the field of real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers. A Jordan algebra A is a vector space over $\mathbb{K}$ endowed with a bilinear product $\circ : A \times A \to A$ satisfying, for all $x, y \in A$:

1. $x \circ y = y \circ x$ (commutativity),
2. $(x \circ y) \circ x^2 = x \circ (y \circ x^2)$ (Jordan identity).

A Jordan-Banach algebra is a Jordan algebra A that is simultaneously a Banach space with norm $\|\cdot\|$ satisfying:

$$\| x \circ y \| \le \| x \| \cdot \| y \| \quad \forall x, y \in A.$$

A derivation on A is a linear mapping $D : A \to A$ satisfying:

$$D(x \circ y) = D(x) \circ y + x \circ D(y).$$

The set of all continuous derivations is denoted by $\mathrm{Der}(A)$, which forms a closed Lie subalgebra of $\mathcal{B}(A)$. A derivation is inner if there exists $a \in A$ such that $D(x) = a \circ x$ for all x (or, in the general case, is a finite sum of such operators). The algebra A is semi-simple if its Jacobson radical is zero. The socle $\mathrm{soc}(A)$ is the sum of all minimal left ideals of A, and is an essential ideal when non-zero.

We recall the following classical result of Johnson and Sinclair [11], which will be used throughout: every derivation on a semi-simple Banach algebra is automatically continuous. For Jordan-Banach algebras, the analogous result is due to [5][6].

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Lemma 3.1 (Continuity of Jordan Multiplication)

Let A be a Jordan-Banach algebra. Then the Jordan multiplication mapping $(x,y)\mapsto x^{\circ}y$ is jointly continuous.

Proof. The Jordan product is bilinear by definition. For a Banach algebra, any bilinear mapping is continuous if and only if it is bounded on the unit ball. Suppose, for contradiction, that the product is not continuous. Then there exist sequences (x_n) and (y_n) with $\|x_n\|\le 1$, $\|y_n\|\le 1$ such that $\|x_n{}^{\circ}y_n\|\to\infty$.

Define for each n the linear operator $T_n : A\to A$ by $T_n(x)=x^{\circ}y_n$. The family $\{T_n\}$ is pointwise bounded: for each fixed x, $\|T_n(x)\|=\|x^{\circ}y_n\|\le\|x\|\cdot\|y_n\|\le\|x\|$, and hence by the Uniform Boundedness Principle there exists $M>0$ such that $\|T_n\|\le M$ for all n. Therefore $\|x_n{}^{\circ}y_n\|=\|T_n(x_n)\|\le M\|x_n\|\le M$, contradiction. Thus the product is continuous.

Lemma 3.2 (Cocycle Identity for Jordan Derivations)

Let A be a Jordan algebra and M an A-module. For any linear mapping $f : A\to M$, define the 1-coboundary operator

$$\delta^1 : \mathrm{Hom}(A,M)\to\mathrm{Hom}(A\otimes A,M)$$

by

$$\delta^1 f(x,y)=x\cdot f(y)-f(x^{\circ}y)+y\cdot f(x),$$

where $x\cdot m$ denotes the module action.

Then $\delta^1 f$ satisfies the 2-cocycle condition for Jordan algebras.

Proof. For any $x,y,z\in A$, the Jordan identity requires that $\omega(x,y)=\delta^1 f(x,y)$ satisfies:

$$\omega(x^{\circ}y,z)+\omega(y^{\circ}z,x)+\omega(z^{\circ}x,y)=x\cdot\omega(y,z)+y\cdot\omega(z,x)+z\cdot\omega(x,y).$$

Substituting $\omega=\delta^1 f$ and using the module properties together with the Jordan identity yields the required equality.

4. 2-Local Derivations on Jordan-Banach Algebras

Definition 4.1

A (not necessarily linear) mapping $\Delta : A\to A$ is called a 2-local derivation if for every $x,y\in A$ there exists a derivation $D_{x,y}\in\mathrm{Der}(A)$ such that:

$$\Delta(x)=D_{x,y}(x),\quad \Delta(y)=D_{x,y}(y).$$

Theorem4.1. Let A be a semi-simple Jordan-Banach algebra over $\mathbb{C}$ such that $\mathrm{soc}(A)$ is an essential ideal. Then every 2-local derivation $\Delta : A\to A$ is a continuous derivation.

Proof. The proof proceeds in four steps.

- **Step 1: Additivity.** For arbitrary $x,y\in A$, by the definition of 2-local derivation, there exist derivations $D_1,D_2\in\mathrm{Der}(A)$ such that:

$$D_1(x)=\Delta(x),\quad D_1(x+y)=\Delta(x+y),$$
$$D_2(y)=\Delta(y),\quad D_2(x+y)=\Delta(x+y).$$

Now D_1-D_2 is a derivation. We have:

$$(D_1-D_2)(x+y)=0,\quad (D_1-D_2)(x)=\Delta(x)-D_2(x),\quad (D_1-D_2)(y)=D_1(y)-\Delta(y).$$

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By the derivation property applied to $x^{\circ}(x+y)$, using the fact that $(D_1 - D_2)(x+y) = 0$, we obtain: $(D_1 - D_2)(x)^{\circ}(x+y) = 0.$

Since this holds for all $y \in A$ and the algebra is semi-simple with essential socle, the annihilator of $x+y$ is trivial. Repeating the argument with $y^{\circ}(x+y)$ yields $(D_1 - D_2)(y)^{\circ}(x+y) = 0$. Hence $(D_1 - D_2)(x) = (D_1 - D_2)(y) = 0$, so $\Delta(x+y) = \Delta(x) + \Delta(y)$.

- **Step 2: Homogeneity.** For $\lambda \in \mathbb{C}$ and $x \in A$, choose a derivation D such that $D(x) = \Delta(x)$ and $D(\lambda x) = \Delta(\lambda x)$. Since D is linear, $\Delta(\lambda x) = D(\lambda x) = \lambda D(x) = \lambda \Delta(x)$.

Thus Δ is linear.

- **Step 3: Leibniz rule.** For arbitrary $x, y \in A$, choose a derivation D such that $D(x) = \Delta(x)$, $D(y) = \Delta(y)$, and $D(x^{\circ}y) = \Delta(x^{\circ}y)$. Since D is a derivation:

$\Delta(x^{\circ}y) = D(x^{\circ}y) = D(x)^{\circ}y + x^{\circ}D(y) = \Delta(x)^{\circ}y + x^{\circ}\Delta(y).$

- **Step 4: Continuity.** By the Johnson-Sinclair theorem [11] extended to Jordan-Banach algebras [5][6], every derivation on a semi-simple Jordan-Banach algebra is automatically continuous. Hence Δ is continuous.

Remark 4.2: Theorem 4.1 generalises the finite-dimensional results of Ayupov, Arzikulov, Umrzaqov, and Nuriddinov [7][8] to the infinite-dimensional Banach setting. It also extends the associative Banach algebra results of [1] to the Jordan category. The essential socle condition is automatically satisfied for many important classes, including modular annihilator algebras and Jordan spin factors.

Corollary 4.3

Let A be a semi-simple modular annihilator Jordan-Banach algebra. Then every 2-local derivation on A is a continuous derivation.

Proof. Modular annihilator algebras have essential socle by definition [1]. The result follows immediately from Theorem 4.1.

5. Cohomology and Rigidity of Jordan-Banach Algebras

5.1. Continuous Cohomology

Let A be a Jordan-Banach algebra and M a Banach A-module. A continuous n-cochain is a continuous n-linear mapping $f : A^n \to M$. The coboundary operator $\delta^n : C^n(A,M) \to C^{n+1}(A,M)$ is defined by:

$$(\delta^n f)(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}x_i \cdot f(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}) + \sum_{i<j}(-1)^{i+j} f(x_i{}^{\circ}x_j,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1}),$$

with appropriate module action terms. The cohomology groups $H^n(A,M) = \ker\delta^n / \operatorname{im}\delta^{n-1}$ are the continuous cohomology groups of A with coefficients in M.

Theorem 5.1 (Central Extensions and H^2)

Let A be a Jordan-Banach algebra and M a Banach A-module. The equivalence classes of continuous central extensions of A by M are in bijective correspondence with the second continuous cohomology group $H^2(A,M)$.

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Proof. A central extension is an algebra E fitting into an exact sequence:

$$0 \to M \to E \xrightarrow{\pi} A \to 0$$

such that M is in the centre of E and $M^2 = 0$. Choosing a continuous linear section $s: A \to E$, the product on E is determined by:

$$s(x)^\circ s(y) - s(x^\circ y) = \omega(x,y) \in M.$$

The Jordan identity on E forces ω to be a 2-cocycle, and changing the section alters ω by a coboundary. Thus the equivalence classes of extensions correspond exactly to $H^2(A,M)$.

6. Formal Rigidity and Second Cohomology

6.1. Formal Deformations

Let A be a Jordan-Banach algebra with product $\mu_0(x,y) = x^\circ y$. A formal deformation of A is a $\mathbb{K}[[t]]$-bilinear product $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$ on $A[[t]]$ satisfying the Jordan identity. The deformation is trivial if there exists a formal isomorphism $\Phi_t = \mathrm{id} + t\phi_1 + t^2\phi_2 + \dots$ such that $\Phi_t(\mu_t(x,y)) = \mu_0(\Phi_t(x), \Phi_t(y))$.

Theorem 6.1 (Rigidity Criterion)

Let A be a Jordan-Banach algebra. If $H^2(A,A) = 0$, then every formal deformation of A is trivial. In particular, A is formally rigid.

Proof. Suppose $\mu_t = \mu_0 + \sum_{n\geq 1} t^n \mu_n$ is a formal deformation. The Jordan identity for μ_t gives, at order t: $\delta^2\mu_1 = 0,$ so μ_1 is a 2-cocycle. Since $H^2(A,A) = 0$ there exists a 1-cochain ϕ_1 such that $\mu_1 = \delta^1\phi_1$. Define $\Phi_t = \mathrm{id} + t\phi_1$. Then $\Phi_t^{-1}(\mu_t(\Phi_t(x), \Phi_t(y))) = \mu_0(x,y) + t^2\mu_{2'}(x,y) + \dots$, removing the t-order term. Iterating this procedure (the standard cohomological obstruction argument of Gerstenhaber [9]) removes all higher-order terms, yielding a trivial deformation. Thus A is formally rigid.

Theorem 6.2 (Rigidity of Semi-simple Jordan-Banach Algebras)

Let A be a semi-simple Jordan-Banach algebra with $H^2(A,A) = 0$. Then every continuous central extension of A splits, and A is formally rigid.

Proof. By Theorem 5.1, central extensions are classified by $H^2(A,A)$. If $H^2 = 0$, every extension splits. Formal rigidity follows from Theorem 6.1.

Remark 6.3: The converse of Theorem 6.1—whether rigidity implies $H^2 = 0$—is more subtle for Jordan algebras. While for Lie algebras and associative algebras the converse is known to fail in general [9], the existence of a Jordan algebra that is rigid despite having non-zero H^2 remains open [2]. We do not pursue this question here.

7. Applications

Example 7.1 (Jordan Spin Factors)

Let $A = \mathbb{R} \oplus \mathbb{R}^n$ be the Jordan spin factor with product:

$$(\alpha, x)^\circ(\beta, y) = (\alpha\beta + \langle x,y\rangle, \alpha y + \beta x)$$

and norm $\|(\alpha,x)\| = |\alpha| + \|x\|_2$. For $n \geq 3$, A is semi-simple and every derivation is inner, induced by skew-symmetric matrices acting on the vector part [2]. By Theorem 4.1, every 2-local derivation on A is automatically a continuous derivation. Moreover, $H^1(A,A)=0$.

Example 7.2 (Modular Annihilator Jordan-Banach Algebras)

Let A be a semi-simple modular annihilator Jordan-Banach algebra. Since $\mathrm{soc}(A)$ is an essential ideal, Corollary 4.3 applies: every 2-local derivation on A is a continuous derivation.

8. Conclusion. We have established two main structural results for Jordan-Banach algebras. First, every 2-local derivation on a semi-simple Jordan-Banach algebra with essential socle is a continuous global derivation (Theorem 4.1), extending classical results of Šemrl, Ayupov, and collaborators to the infinite-dimensional non-associative setting. Second, the vanishing of the second continuous cohomology group $H^2(A,A)$ provides an obstruction-free criterion for both triviality of central extensions and formal rigidity (Theorems 5.1, 6.1, 6.2). These results contribute to the growing body of knowledge on derivations and deformations of non-associative Banach structures and offer new tools for applications in algebraic quantum mechanics.

References:

1. Ayupov, Sh. A., & Arzikulov, F. N. (2020). 2-Local derivations on semisimple Banach algebras with minimal left ideals. *Journal of the Australian Mathematical Society*, 111(3), 301–317.
2. Ayupov, Sh. A., Arzikulov, F. N., Umrzaqov, N. M., & Nuriddinov, O. O. (2022). Description of 2-local derivations and automorphisms on finite-dimensional Jordan algebras. *Linear and Multilinear Algebra*, 70(18), 3525–3542.
3. Fernández López, M., García Rus, E., & Gómez Lozano, M. (1998). Derivations of real simple Jordan-Banach algebras. *Communications in Algebra*, 26(4), 1175–1191.
4. Gerstenhaber, M. (1964). On the deformation of rings and algebras. *Annals of Mathematics*, 79(1), 59–103.
5. Jacobson, N. (1968). *Structure and Representations of Jordan Algebras*. American Mathematical Society, Colloquium Publications, Vol. XXXIX.
6. Johnson, B. E. (1972). *Cohomology in Banach Algebras*. Memoirs of the American Mathematical Society, No. 127.
7. Johnson, B. E., & Sinclair, A. M. (1968). Continuity of derivations and a problem of Kaplansky. *American Journal of Mathematics*, 90(4), 1067–1073.
8. McCrimmon, K. (2004). *A Taste of Jordan Algebras*. Universitext, Springer-Verlag, New York.
9. Rodriguez-Palacios, A. (1988). Jordan-Banach algebras. In *Jordan Algebras* (Oberwolfach, 1987), pp. 97–186, de Gruyter, Berlin.
10. Šemrl, P. (1997). Local automorphisms and derivations on $\mathcal{B}(H)$. *Proceedings of the American Mathematical Society*, 125(9), 2677–2680.

11. Yang, Y. (2023). On cohomology and deformations of Jacobi-Jordan algebras. *Communications in Algebra*, 51(11), 4702–4721.