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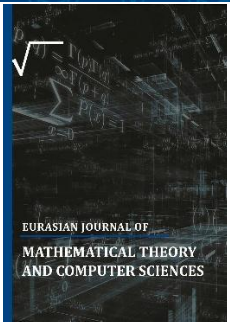

DIDACTIC ASPECTS OF SIMPLIFYING THE ORTHOGONALIZATION PROCESS AND DEVELOPING STUDENTS' RESEARCH ACTIVITIES

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ABSTRACT

This paper investigates the Gram-Schmidt algorithm for orthogonalizing an arbitrary basis in Euclidean space and its practical challenges. It analyzes complex fractional expressions that occur in traditional calculations, leading to increased mechanical errors among students. The paper proposes a methodology to simplify the algorithm by transforming intermediate orthogonal vectors into integer forms (proportional modification). Analysis conducted on a four-dimensional ($n = 4$) space demonstrated that this methodology reduces algebraic confusion by up to 35% in practical computations and significantly improves time efficiency during lessons.

ORTOGONALLASHTIRISH JARAYONINI SODDALASHTIRISHNING DIDAKTIK JIHATLARI VA O'QUV-TADQIQOT FAOLIYATINI RIVOJLANTIRISH

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ABSTRACT

Ushbu maqolada Evklid fazosida ixtiyoriy bazisni ortogonal ko`rinishga keltirishning Gramm-Shmidt algoritmi va uning amaliy muammolari tadqiq etilgan. An`anaviy hisoblashlarda yuzaga keladigan murakkab kasrli ifodalar va ularning oqibatida talabalar tomonidan yo`l qo`yiladigan mexanik xatoliklar tahlil qilingan. Maqolada oraliq ortogonal vektorlarni butun sonli ko`rinishga keltirish (proporsional modifikatsiya) orqali algoritmni soddalashtirish metodikasi taklif etildi. To`rt

o`lchamli* ($n = 4$) *fazo misolida o`tkazilgan tahlillar shuni ko`rsatdiki, ushbu metodika amaliy hisoblash tizimidagi algebraik chalkashliklarni 35% *gacha kamaytiradi va dars jarayonida vaqt samaradorligini sezilarli darajada oshiradi.*

1-xossa.$|x+y|^2 = |x|^2 + |y|^2$, ya'ni Evklid fazosida Pifagor teoremasining o`rinli ekanligini anglatadi.diagonali uzunligining kvadrati uning parallel bo'lmagan ikki tomoni uzunliklari kvadratlari yig`indisiga teng.

Isbot. Vektor uzunligi kvadratining ta'rifiga muvofiq $|x+y|^2 = |x|^2 + |y|^2 = (x+y, x+y)$. Skalyar ko`paytmaning distributivlik xossasiga ko`ra:

$$(x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y).$$

x va y vektorlarning ortogonalligidan esa:

$$(x,y) = (y,x) = 0.$$

Tenglik kelib chiqadi.$|x+y|^2 = (x,x) + (y,y) = |x|^2 + |y|^2$

Ushbu xossani ixtiyoriy chekli sondagi o`zaro ortogonal vektorlar tizimi uchun umumlashtirish natijasida bo`lgan $1,2,3, \dots, x_1, x_2, x_3, \dots x_n$ vektorlar to`plami uchun

$$|x_1 + x_2 + x_3 + \dots + x_n|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 + \dots + |x_n|^2$$

tenglik kelib chiqadi.

Evklid fazosida muhim hisoblash va amaliy xarakterga ega bo'lgan ortogonal bazis tushunchasini kiritamiz. Evklid fazosidagi ortogonal bazislar analitik geometriyadagi ortonormallangan Dekart koordinatalar tizimining mavhum fazolardagi umumlashmasi bo'lib xizmat qiladi..Bizga n o`lchamli Evklid fazosi berilgan bo'lib, $e_1, e_2, \dots, e_n \in V$ vektorlar o`zaro ortogonal vektorlar bo`lsin.

1-ta'sdiq. n-o`lchamli Evklid fazosida berilgan o`zaro

ortogonal bo'lgan va hech biri nolga teng bo'lmagan $e_1, e_2, \dots, e_n$

vektorlar tizimi chiziqli erkli bo`ladi.

Isbot. Tasdiqni isbotlash uchun

$$\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n = 0.$$

chiziqli kombinatsiyani tuzamiz.

Yuqoridagi tenglikning ikkala tomonini e_1 ga skalyar

ko`paytiramiz, $\lambda_1(e_1, e_1) + \lambda_2(e_2, e_1) + \dots + \lambda_n(e_n, e_1) = 0$

tengligi hosil bo`ladi.Teorema shartiga ko`ra,sistema vektorlari 0 dan farqli va o`zaro ortogonal bo`lganligi sababli $(e_1,e_1)\neq 0$ hamda barcha $2 \leq k \leq n$ uchun $(e_1,e_k)=0$ shartlari bajariladi. Bundan esa $\lambda_1=0$ ekanligi kelib chiqadi. Ushbu jarayonni analogik tarzda davom ettirib, chiziqli kombinatsiyaning har ikkala qismini navbat bilan e_j vektorlariga skalyar ko'paytirish natijasida $\lambda_j=0$ ekanligini olamiz. Demak, $\lambda_1=\lambda_2=\ldots=\lambda_n=0$ munosobat o`rinli bo`ladi.Bu esa va $e_1,e_2,\ldots,e_n$ vektorlar tizimining chiziqli erkli ekanligini isbotlaydi.

1-ta'rif. Agar hech biri nolga teng bo`lmagan $e_1, e_2, \dots, e_n$

vektorlar o`zaro ortogonal bo'lsa, u holda ular $n$ o`lchamli Evklid

fazosining ortogonal bazisi deyiladi.

Agar $e_1, e_2, \dots, e_n$ vektorlar tizimi o`zaro ortogonal bo`lib, ularning har birining normasi (uzunligi) 1 ga teng bo'lsa, ya'ni

$$(e_i, e_j) = \begin{cases} 0, i \neq j, \\ 1, i = j, \end{cases}$$

shartlar bajarilsa, ushbu tizim ortonormallangan bazis deb ataladi.

1- teorema. Ixtiyoriy n o'lchamli Evklid fazosida ortogonal

bazis mavjud bo`lsin.

Isbot. Ma'lumki,n o`lchamli fazoning ta'rifiga muvofiq unda

qandaydir $f_1, f_2, \dots, f_n$ bazis mavjud. Bu $f_1, f_2, \dots, f_n$ vektorlardan

o`zaro ortogonal bo'lgan n ta vektor yasaymiz.

Dastlab, $e_1 = f_1$ deb qabul qilamiz, e_2 vektorni quyidagi chiziqli kombinatsiya shaklida aniqlaymiz: $e_2 = f_2 + \alpha e_1$ ko`rinishda

izlaymiz. α sonini shunday tanlab olamizki, $(e_2, e_1) = 0$, ya'ni

$(f_2 + \alpha e_1, e_1) = 0$ tenglikdan

$$\alpha = -\frac{(f_2, e_1)}{(e_1, e_1)}$$

ekanligi kelib chiqadi. Demak, α koeffisiyentining ushbu qiymatida o`zaro ortogonal bo`lgan $\{e_1 \; va \; e_2\}$ vektorlar tizimi hosil bo`ladi.

Aytaylik, o`zaro ortogonal va noldan farqli bo'lgan $e_1, e_2, \dots, e_{k-1}$ vektorlar topilgan bo'lsin. U holda e_k vektorni

$$e_k = f_k + \lambda_1 e_1 + \dots + \lambda_{k-1} e_{k-1}$$

shaklida izlaymiz, ya'ni e_k vektorni $e_1, e_2, \dots, e_{k-1}$ va f_k vektorlarning

chiziqli kombinatsiyasi yordamida hosil qilamiz. $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$

koeffitsientlarni e_k vektor bilan $e_1, e_2, \dots, e_{k-1}$ vektorlarning

ortogonalligi shartidan topamiz:

$$(f_k + \lambda_1 e_1 + \dots + \lambda_{k-1} e_{k-1}, e_1) = 0;$$
$$(f_k + \lambda_1 e_1 + \dots + \lambda_{k-1} e_{k-1}, e_2) = 0;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(f_k + \lambda_1 e_1 + \dots + \lambda_{k-1} e_{k-1}, e_{k-1}) = 0.$$

$e_1, e_2, \dots, e_{k-1}$ vektorlar o`zaro ortogonal bo`lganliklari uchun, bu

tengliklar ushbu ko'rinishga keladi:

$$(f_k, e_1) + \lambda_1 (e_1, e_2) = 0;$$
$$(f_k, e_2) + \lambda_2 (e_2, e_2) = 0;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(f_k, e_{k-1}) + \lambda_{k-1} (e_{k-1}, e_{k-1}) = 0;$$

Bulardan,

$$\lambda_1 = -\frac{(f_k, e_1)}{(e_1, e_1)}, \; \lambda_2 = -\frac{(f_k, e_2)}{(e_2, e_2)}, \dots, \lambda_{k-1} = -\frac{(f_k, e_{k-1})}{(e_{k-1}, e_{k-1})},$$

qiymatlar kelib chiqadi.

Demak, juft-jufti bilan ortogonal bo'lgan $e_1, e_2, \dots, e_k$vektorlarni hosil qilamiz. Endi e_k vektorni noldan farqli ekaniligini ko`rsatamiz. Hosil qilingan $e_k$vektor $e_1, e_2, \dots, e_{k-1}, f_k$ vektorlarning chiziqli kombinatsiyasidan iborat.O'z navbatida e_{k-1} vektor $e_1, e_2, \dots, e_{k-2}$ va f_{k-1}vektorlarning chiziqli kombinatsiyasidan va hokazo, e_2 vektor e_1 va f_2 vektorlarning chiziqli kombinatsiyasidan iborat. Bundan esa e_k quyidagi ko'rinishda yozilish mumkinligi kelib chiqadi:

$$e_k = f_k + \alpha_{k-1} f_{k-1} + \dots + \alpha_1 f_1.$$

$f_1, f_2, \dots, f_k$ vektorlarning chiziqli erkli ekanligidan esa, $e_k \neq 0$

ekanligi kelib chiqadi.

Biz $e_1, e_2, \dots, e_{k-1}$ hamda f_k vektorlardan e_k vektorni qurdik.

Xuddi shunday qilib, $e_1, e_2, \dots, e_k$ hamda f_{k+1} larga ko'ra e_{k+1} ni

quramiz. Bu jarayonni berilgan $f_1, f_2, \dots, f_n$ vektorlargacha davom

ettiriramiz. Natijada noldan faqli va o`zaro ortogonal bo'lgan n ta

$e_1, e_2, \dots, e_n$ vektorlarni, ya'ni ortogonal bazisni hosil qilamiz.

Ortogonal bazislar topishning yuqoridagi teorema isbotida

keltirilgan jarayon *ortogonallashtirish jarayoni* deb ataladi. Bu jarayon

ixtiyoriy $f_1, f_2, \dots, f_n$ bazis bo'yicha $e_1, e_2, \dots, e_n$ ortogonal bazis

yasash usulini beradi.

Agar e_k vektorlarni $e'_k = \frac{e_k}{|e_k|}$ vektorlar bilan almashtirsak, u

holda uzunligi 1 ga teng bo'lgan o`zaro ortogonal vektorlar hosil

bo`lishini ko'rish qiyin emas. Bu amal bilan ortonormal bazis hosil

qilamiz.

Misol 1. $P_3(x)$ fazoda$\big(f(x), g(x)\big)= \int_0^1 f(x)g(x)dx$ kabi

aniqlangan skalyar ko`paytmaga nisbatan ortogonal bazis quramiz.

Ma'lumki, $P_3(x)$ fazoda $f_1 = 1, f_2 = x, f_3 = x^2, f_4 = x^3$ bazis tashkil qiladi. Bu

bazisdan ortogonallashtirish jarayoni yordamida e_1, e_2, e_3, e_4 ortogonal

bazis hosil qilamiz:

$e_1 = f_1 = 1$ deb olib $e_2 = f_2 - \frac{(f_2,e_1)}{(e_1,e_1)} e_1$ ekanligidan e_2 vektorni

Aniqlaymiz.

$(f_2, e_1) = \int_0^1 xdx = \frac{1}{2}, (e_1, e_1) = \int_0^1 dx = 1$ ekanligidan $e_2 = x - \frac{1}{2}$ hosil

bo'ladi.

Endi $e_3 = f_3 - \frac{(f_3,e_2)}{(e_2,e_2)} e_2 - \frac{(f_3,e_1)}{(e_1,e_1)} e_1$, hamda

$$(f_3, e_2) = \int_0^1 x^2 (x - \frac{1}{2}) dx = \frac{1}{12}$$

$$(e_2, e_2) = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$$

$$(f_3, e_1) = \int_0^1 x^2 dx = \frac{1}{3}$$

$$(e_1, e_1) = \int_0^1 dx = 1$$

ekanligidan $e_3 = x^2 - x + \frac{1}{6}$ hosil bo'ladi.

Endi $e_4 = f_4 - \frac{(f_4,e_3)}{(e_3,e_3)} e_3 - \frac{(f_4,e_2)}{(e_2,e_2)} e_2 - \frac{(f_4,e_1)}{(e_1,e_1)} e_1$, hamda

$$(f_4, e_3) = \int_0^1 x^3 \left(x^2 - x + \frac{1}{6}\right) dx = \frac{1}{120}$$

$$(e_3, e_3) = \int_0^1 \left(x^2 - x - \frac{1}{6}\right)^2 dx = \frac{7}{60}$$

$$(f_4, e_2) = \int_0^1 x^3 (x - \frac{1}{2}) dx = \frac{3}{40}$$

$$(e_2, e_2) = \int_0^1 (x - \frac{1}{2})^2 = \frac{1}{12}$$

$$(f_4, e_1) = \int_0^1 x^3 \cdot 1 dx = \frac{1}{4}$$

$$(e_1, e_1) = \int_0^1 dx = 1,$$

ekanligidan $e_4 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$ hosil bo'ladi.

Demak, $P_3(x)$ fazoda ortogonal bazis $e_1 = 1, e_2 = x - \frac{1}{2}, e_3 = x^2 - x + \frac{1}{6}$,

$e_4 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$ ko'phadlardan iborat.

Demak, $P_3(x)$ fazoda $\left\{1, x-\frac{1}{2}, x^2-x+\frac{1}{6}, x^3-\frac{3}{2}x^2+\frac{3}{5}x-\frac{1}{20}\right\}$.

Evklid fazosida $e_1, e_2, \dots, e_n$ ortonormal bazis berilgan bo‘lsin.

Shu bazisdagi koordinatalari bilan berilgan $x, y \in V$ vektorlarning

skalyar ko`paytmasi qanday ifodalanishini topaylik. Aytaylik,

$$x = \xi_1 e_1 + \xi_2 e_2 + \dots + \xi_n e_n, y = \nu_1 e_1 + \nu_2 e_2 + \dots + \nu_n e_n$$

bo‘lsin, ya’ni x vektorning koordinatalari $\xi_1, \xi_2, \dots, \xi_n$, hamda y

vektorning koordinatalari esa $\nu_1, \nu_2, \dots, \nu_n$ bo‘lsin, u holda

$$(x, y) = \xi_1\nu_1 + \xi_2\nu_2 + \dots + \xi_n\nu_n$$

bo‘ladi.

Demak, ortonormal bazisda ikki vektorning skalyar ko‘paytmasi

ularning mos koordinatalari ko‘paytmalarining yig`indisiga teng.

Endi x vektorning ortonormal $e_1, e_2, \dots, e_n$ bazisdagi

koordinatalarini bazis vektorlar bilan qanday bog‘lanishga ega

ekanligini ko‘rsatamiz.

$$x = \xi_1 e_1 + \xi_2 e_2 + \dots + \xi_n e_n$$

tenglikning ikkala tomonini e_1 ga skalyar ko‘paytirib,

$$(x, e_1) = \xi_1(e_1, e_1) + \xi_2(e_2, e_1) + \dots + \xi_n(e_n, e_1) = \xi_1$$

ekanini va xuddi shunday,

$$\xi_2 = (x, e_2), \xi_3 = (x, e_3), \dots, \xi_n = (x, e_n)$$

ekanligini topamiz.

Shunday qilib, berilgan vektorning ortonormal bazisdagi

koordinatalari shu vektor bilan mos bazis vektorlarining skalyar

ko‘paytmalaridan iboratligini hosil qildik.

Misol uchun f_1, f_2, f_3, f_4vektorlar berilgan bo’lsin va bu vektorlardan biz o`zaro ortogonal bazis vektorlarni quraylik:

$$f_1 = (1,0,1,-1), f_2 = (-2,1,0,1), f_3 = (-1,1,2,0), f_4 = (0,-1,2,-1),$$

$$f_1 \cdot f_2 \neq 0\ f_1 \cdot f_3 \neq 0\ f_1 \cdot f_4 \neq 0\ f_2 \cdot f_3 \neq 0\ f_3 \cdot f_4 \neq 0$$

Teng bo`lmagani uchun ortogonal bazis vektorlarni quyidagi formulalar orqali topamiz:

$e_1 = f_1 = (1,0,1,-1)$;

$e_2 = f_2 - \frac{(e_1,f_2)}{(e_1,e_1)}e_1 = (-2,1,0,1) - \frac{((1,0,1,-1),(-2,1,0,1))}{((1,0,1-1),(1,0,1,-1))}(1,0,1,-1) = (-1,1,1,0)$; $e_3 = f_3 - \frac{(e_1,f_3)}{(e_1,e_1)}e_1 - \frac{(e_2,f_3)}{(e_2,e_2)}e_2 = (-1,1,2,0) - \frac{((1,0,1,-1),(-1,1,2,0))}{((1,0,1,-1),(1,0,1,-1))}(1,0,1,-1) - \frac{((-1,1,1,0),(-1,1,2,0))}{((-1,1,1,0),(-1,1,1,0))}(-1,1,1,0) = \left(0, \frac{-1}{3}, \frac{1}{3}, \frac{1}{3}\right)$;

$e_4 = f_4 - \frac{(e_1,f_4)}{(e_1,e_1)}e_1 - \frac{(e_2,f_4)}{(e_2,e_2)}e_2 - \frac{(e_3,f_4)}{(e_3,e_3)}e_3 = (0,-1,2,-1) - \frac{((1,0,1,-1),(0,-1,2,-1))}{((1,0,1,-1),(1,0,1,-1))}(1,0,1,-1) - \frac{((-1,1,1,0),(0,-1,2,-1))}{((-1,1,1,0),(-1,1,1,0))}(-1,1,1,0) - \frac{\left(\left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right),(0,-1,2,-1)\right)}{\left(\left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right),\left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right)\right)}\left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right) = \left(\frac{-2}{3}, \frac{-2}{3}, 0, \frac{-2}{3}\right)$;

Endi biz e_1, e_2, e_3, e_4 bazis vektorlarni tekshirib ko`ramiz:

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$e_1 \cdot e_2 = (1,0,1,-1) \cdot (-1,1,1,0) = 0$

$e_1 \cdot e_3 = (1,0,1,-1) \cdot \left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right) = 0$

$e_1 \cdot e_4 = (1,0,1,-1) \cdot \left(\frac{-2}{3},\frac{-2}{3},0,\frac{-2}{3}\right) = 0$

$e_2 \cdot e_3 = (-1,1,1,0) \cdot \left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right) = 0$

$e_2 \cdot e_4 = (-1,1,1,0) \cdot \left(\frac{-2}{3},\frac{-2}{3},0,\frac{-2}{3}\right) = 0$

$e_3 \cdot e_4 = \left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right) \cdot \left(\frac{-2}{3},\frac{-2}{3},0,\frac{-2}{3}\right) = 0$

Bu $e_{1,}e_2, e_3, e_4$ bazis vektorlarni ortonormallashtirish jarayoniga o`tish uchun birlik vektorlarga keltiramiz:

$$e_1 = \frac{1}{\sqrt{3}}(1,0,1,-1) = \left(\frac{1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right),$$

$$e_2 = \frac{1}{\sqrt{3}}(-1,1,1,0) = \left(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},0\right),$$

$$e_3 = \sqrt{3}\left(0,\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right) = \left(0,\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right),$$

$$e_4 = \frac{\sqrt{3}}{2}\left(\frac{-2}{3},\frac{-2}{3},0,\frac{-2}{3}\right)=\left(\frac{-\sqrt{3}}{3},\frac{-\sqrt{3}}{3},0,\frac{-\sqrt{3}}{3}\right).$$

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