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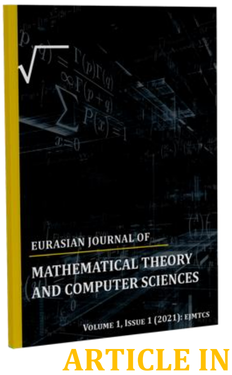


# PROOF OF LEE-YANG THEOREM FOR N=2 CASES

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ABSTRACT

*There are many interrelated theories in mathematical physics and complex analysis, among which the Lee-Yang theorem has a special place. This theorem plays an important role in statistical physics and the theory of phase transitions. By analyzing the roots of polynomials in the complex plane, it allows to study the behavior of physical systems. Also, this theorem is considered as a theoretical basis of thermodynamic limit and state transitions in physics. The main importance of the Lee-Yang theorem is that it geometrically limits the roots of polynomials of complex variables and helps to determine the properties of statistical systems.*

*The main goal of this study is to provide a detailed mathematical proof of the Lee-Yang theorem for the cases n=2 and n=3 and to study its significance in relation to statistical physics.*

# LEE-YANG TEOREMASINI N=2 HOLAT UCHUN ISBOTLASH

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ABSTRACT

*Matematik fizika va kompleks tahlilda o'zaro bog'liq ko'plab nazariyalar mavjud bo'lib, ular orasida Li-Yang teoremasi alohida o'rin tutadi. Bu teorema statistik fizika va fazaviy o'tishlar nazariyasida muhim rol o'ynaydi. Polinomlarning kompleks tekislikdagi ildizlarini tahlil qilish orqali fizik tizimlarning xatti-harakatini o'rganish imkonini beradi. Shuningdek, bu teorema fizikada termodinamik limit va holat o'tishlarining nazariy asosi sifatida ko'riladi. Li-Yang teoremasining asosiy*

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*ahamiyati, u kompleks o‘zgaruvchilar polinomining ildizlarini geometrik jihatdan chegaralaydi va statistik tizimlarning xususiyatlarini aniqlashga yordam beradi. Ushbu tadqiqotning asosiy maqsadi Li-Yang teoremasining n=2 va n=3 hollar uchun matematik isbotini batafsil keltirish va uning statistik fizika bilan bog‘liq ahamiyatini o‘rganishdan iboratdir.*

Kirish: Li-Yang teoremasi bir nechta shakllarda ifodalangan. Asl shakli polinomlarning ildizlarini kompleks tekislikda tahlil qilishga qaratilgan, lekin keyinchalik bu teorema turli tizimlarga, masalan, o‘zgaruvchan o‘lchovlar yoki tizimdagi o‘zgaruvchan parametrlarga moslashtirilgan.

Teoremaning eng muhim shakli, kompleks polinomlarining ildizlarini modulusiga qarab tashkil qilishdir. Teoremaning ba’zi maxsus holatlari (masalan,n=2 , ) uchun bu polinomlar tizimning xatti-harakatlarini yaxshiroq tushunish mumkin.

Lee – Yang teoremsi :Teorema : (Li Yang teoremasi). Λ={1, ..., n} va $(B_{ij})$ $i \geq 1\,, j \leq n$ , n×n o'lchamdagi simmetrik matritsabo’lsin. Barcha i, j uchun $0 < B_{ij} \leq 1$ , n ta murakkab o'zgaruvchilardan tenglik bo'yicha

$p(z_1, \dots, z_n)$ ko‘phadni aniqlaymiz.

$$p(z_1, \dots, z_n) = \sum_{A \subset \Lambda} z^A C(A)$$

buyerda

$$z^A = \prod_{i \in A} z_i \;\; agar \;\; A \neq \emptyset\,, \qquad z^{\emptyset} = 1$$

Agar $\xi_1, \dots \xi_n \;\in C \quad va\; |\xi_i| \geq 1 \qquad i = 1, \dots, n-1\; lar\; uchun$

$$P(\xi_1, \dots, \xi_n) = 0$$

Agar $|\xi_i| \leq 1$ bo’lsa uxolda yuqoridagi ko’phad

$p(z_1^{-1}, \dots z_n^{-1}) = z_1^{-1} \dots z_n^{-1} p(z_1, \dots, z_n)$

biz bundan quyidagilarni olamiz

$$\xi_1, \dots \xi_n \;\in C \;\; |\xi_i| \leq 1 \quad i = 1, \dots, n-1\; lar\; uchun\; va\;\; p(\xi_1, \dots \xi_n) = 0$$

Quyidagicha funksiya kritaylik

$$\sigma_A : \mathcal{P}(\Lambda) \to \{-1,1\}$$
$$\sigma_A(B) = (-1)^{|A \cap B|}$$

Bundan quyidagilar aniqlangan bo’lsin

$$J : \mathcal{P}(\Lambda) \to R$$
$$J(\{i,j\}) = -\log B_{ij} \;\; agar \;\; i \neq j,$$
$$J(B) = 0\; agar \quad |B| \neq 2$$

Buyerda B⊂Λ lardan biz J(B)≥0 ekanini olamiz va unda harbir A⊂Λ qism to’plamlar uchun quyidagi tenglik o’rinli.

$$\sum_{B \subset \Lambda} \sigma_B(A) J(B) = 2 \sum_{i \in A} \sum_{j \in \Lambda - A} \log B_{ij} - \frac{1}{2} \sum_{i \in \Lambda} \sum_{j \in \Lambda} \log B_{ij}$$

Bu tenglamadan Lee –Yang teoremasi kelib chiqadi.

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**N=2 holi uchun isbot:**Biz yuqoridagi tenglamani n=2 holat uchun ko'rib chiqsak . Λ={1, 2}

Endi biz n=2 holini koʻrib chiqamiz. Bu holda polinom quyidagicha boʻladi:

$$p(z_1, z_2) = C(\emptyset) + C(\{1\})z_1 + C(\{2\})z_2 + C(\{1,2\})z_1 z_2$$

Bu yerda , $C(\emptyset), C(\{1\}), C(\{2\})\ va\ C(\{1,2\})$ koeffitsiyentlar $B_{12}$ orqali aniqlanadi:

$$C(\emptyset) = B_{12}, \qquad C(\{1\}) = B_{12}, \qquad C(\{2\}) = B_{12}\,, \qquad C(\{1,2\}) = B_{12}$$

$C(\emptyset)$— boʻlinmalarning umumiy koeffitsiyenti (barcha oʻzgaruvchilar uchun),

- $C(\{1\})$ — faqat $z_1$ uchun koeffitsiyent,
- $C(\{2\})$— faqat $z_2$ uchun koeffitsiyent,
- $C(\{1,2\})$ — $z_1$ va $z_2$ ning oʻzaro koeffitsiyenti.

Polinomni quyidagicha yozamiz:

$$p(z_1, z_2) = B_{12} + z_1 + z_2 + B_{12} z_1 z_2$$

Ildizlarni tahlil qilamiz. Agar polinomning ildizlari $z_1$ va $z_2$ boʻlsa, quyidagi tenglikdan foydalanamiz:

$$B_{12} + z_1 + z_2 + B_{12} z_1 z_2 = 0$$

Bu tenglikni modulo 1 ga qarab koʻrib chiqamiz. Teoremaning shartlariga koʻra, agar $z_1 \geq 1$ va $z_2 \geq 1$ boʻlsa, polinomning ildizlari ham modulus 1 boʻlishi kerak. Bu isbotning asosiy qismidir: $|z_1| = |z_2| = 1$

**N=2 xolat uchun isbotning yana bir usuli.**

$$Z_2 = -\frac{1 + B_{12}(x + iy)}{B_{12} + x + iy}$$

$$|Z_2| = \left|-\frac{1 + B_{12}(x + iy)}{B_{12} + x + iy}\right| = \frac{|1 + B_{12}(x + iy)|}{|B_{12} + x + iy|}$$

$$|1 + B_{12}(x + iy)| = 1 + B_{12}x + iB_{12}y$$

$$|1 + B_{12}(x + iy)| = \sqrt{(1 + B_{12}x)^2 + (B_{12}y)^2}$$

$$|B_{12} + x + iy| = \sqrt{(B_{12} + x)^2 + y^2}$$

$$\frac{|1 + B_{12}(x + iy)|}{|B_{12} + x + iy|} \geq 1$$

$$|1 + B_{12}(x + iy)| \geq |B_{12} + x + iy|$$

$$(1 + B_{12}x)^2 + (B_{12}y)^2 \geq (B_{12} + x)^2 + y^2$$

$$1 + 2B_{12}x + {B_{12}}^2 x^2 + {B_{12}}^2 y^2 \geq {B_{12}}^2 + 2B_{12}x + x^2 + y^2$$

$$1 + {B_{12}}^2 x^2 + {B_{12}}^2 y^2 \geq {B_{12}}^2 + x^2 + y^2$$

$$1 + {B_{12}}^2 (x^2 + y^2) \geq {B_{12}}^2 + x^2 + y^2$$

$$1 \geq (1 - {B_{12}}^2)(x^2 + y^2)$$

$${B_{12}}^2 \leq 1, yani\ |B_{12}| \leq 1, u\ holda$$

$1 - B^2 \geq 0$ va

$$1 \geq (1 - {B_{12}}^2)(x^2 + y^2)$$

har doim o'rinli bo'ladi, chunki $(x^2+y^2)\geq 0$

Shuning uchun $|Z_2|\geq 1$

**$z_1$ va $z_2$ ni x va y o'zgaruvchilarning bazi qiymatlarida hisoblash**

Agar x=0.3 va y=0.4 bo'lsa $|Z_2|\geq 1$ bo'lishni tekshiramiz

$$|1+B_{12}(x+iy)|\geq|B_{12}+x+iy|$$

Shu tengsizlikga qo'yishdan boshlaylik

$$|1+B_{12}(0.3+i0.4)|\geq|B_{12}+0.3+i0.4|$$

Bundan modulni qiymatini hisoblash orqali

$$(1+B_{12}x)^2+(B_{12}y)^2\geq(B_{12}+x)^2+y^2$$

$$(1+B_{12}0.3)^2+(B_{12}0.4)^2\geq(B_{12}+0.3)^2+(0.4)^2$$

$$1+0.6B_{12}+0.9{B_{12}}^2+0.16{B_{12}}^2\geq{B_{12}}^2+0.6B_{12}+0.9+0.16$$

$$1+0.9{B_{12}}^2+0.16{B_{12}}^2\geq{B_{12}}^2+0.25$$

$$1\geq(1-{B_{12}}^2)0.25$$

Ga kelamiz. Biz $B_{ij}$ ga ham bir nechta qiymat berib tengsizlik o'rinli bo'lishni ko'rishimiz mumkin.

$B_{12}=0.5$ bo'lsa

$$1\geq(1-(0.5)^2)0.25$$

$$1\geq 0.1875$$

$B_{12}=1$ bo'lsa

$$1\geq 0$$

$B_{12}$ , x va y ning qiymatlari uchun $|Z_2|$ ni hisoblaylik

$$B_{12}=0.5\ ,\qquad x=0.3\ \ va\ \ y=0.4$$

$$|Z_2|\geq\frac{(1+B_{12}x)^2+(B_{12}y)^2}{(B_{12}+x)^2+y^2}$$

$$|Z_2|\geq\frac{(1+0.5\times0.3)^2+(0.5\times0.4)^2}{(0.5+0.3)^2+(0.4)^2}$$

$$Z_2\approx 1.7$$

$B_{12}=1$ da $Z_2=1$

$B_{12}=2$ da $Z_2=0.77$

Yuqoridagi natijalardan ko'rnib turbdiki $0<B_{ij}\leq 1\,, x,y<1$bo'ganda $|Z_2|\geq 1$ bo'lar ekan

Li-Yang teoremasi, ehtimollik nazariyasida ishlatiladigan muhim vositalardan biridir. U, xususan, statistik mexanika va kvant mexanikasi kabi sohalarda, polinomlarning ildizlarini o'rganish bilan bog'liq. Ushbu bo'limda, teoremaning matematik asoslari va uning polinomlar bilan bog'liq bo'lgan qismi yanada batafsil ko'rib chiqiladi.

Li-Yang Teoremasining Formulasi

Li-Yang teoremasi polinomlarning ildizlari haqida maxsus xususiyatlar beradi. Teoremaga ko‘ra, agar polinomning har bir koeffitsiyenti maxsus shartlarni bajaradigan bo‘lsa, unda polinomning ildizlari birlik doirasida joylashgan bo‘ladi. Teoremaning umumiy shakli quyidagicha ifodalanadi:

$$p(z_1, z_2, \dots, z_n) = C(\emptyset) + C(\{1\})z_1 + C(\{2\})z_2 + \dots + C(\{n\})z_n$$

Bu yerda, $C(\emptyset), C(\{1\}), C(\{2\}), \dots, C(\{n\})$ , koeffitsiyentlar polinomni tashkil etadi va har bir koeffitsiyentning qiymati ma'lum cheklovlarga ega bo‘ladi. Bu shartlar polinomning ildizlari qanday joylashishini ta'minlaydi.

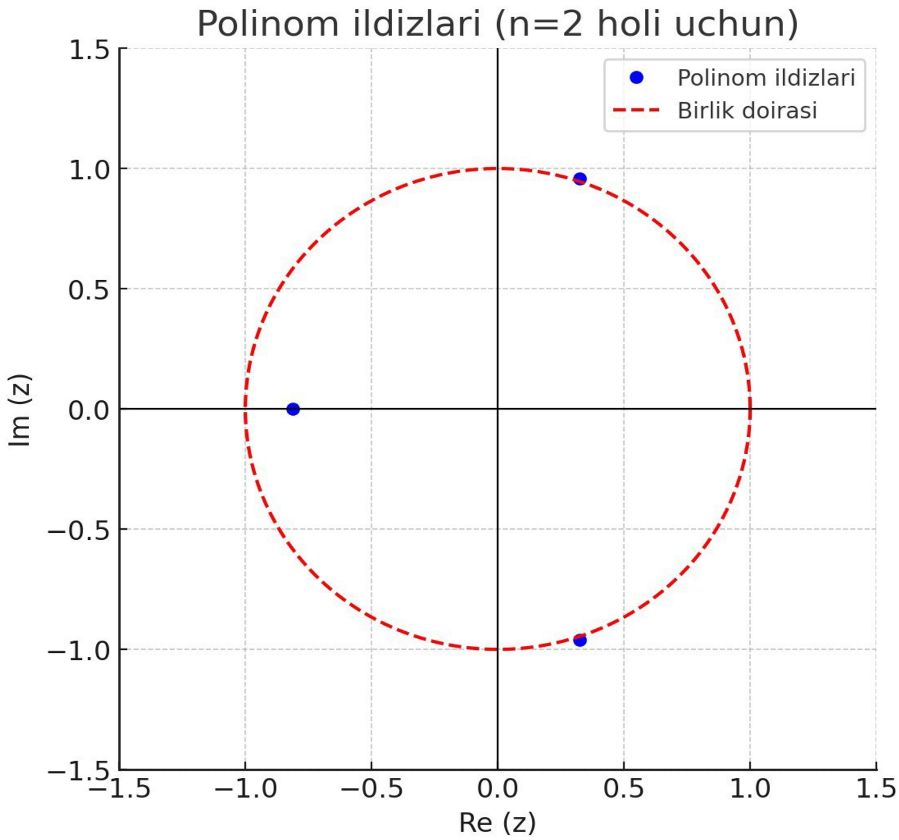
Koeffitsiyentlarning Shartlari

Li-Yang teoremasining koeffitsiyentlariga qo‘yilgan shartlar quyidagi tarzda ifodalanadi:

- C-koeffitsiyentlari 0 dan 1 gacha bo‘lgan qiymatlarni olishi kerak.
- Polinomning ildizlari modulda 1 ga teng bo‘lgan doirada joylashgan bo‘lishi zarur.

Bu shartlar polinomning ildizlarining xususiyatlarini aniqlashda muhim rol o‘ynaydi.

**Xulosa:** Li-Yang teoremasining kuchli tomoni shundaki, u faqat koeffitsiyentlarning cheklangan qiymatlari asosida polinomning ildizlarining joylashuvi haqida aniq natijalar beradi. Bu polinomlarning ildizlarini tahlil qilish orqali, har bir ildizning moduli 1 bo‘lishi kerakligini ko‘rish mumkin. Ya'ni,  bo‘lishi shart. Bu xususiyatlarni hisoblashda, koeffitsiyentlarining modullari 1 dan kichik yoki teng bo‘lishi kerak. Bu koeffitsiyentlar orasidagi bog‘lanishlarning natijasida polinomning ildizlari  bo‘lishi ta’minlanadi.




Mana, n=2 holi uchun polinomning ildizlari grafikda ko‘rsatilgan. Ko‘k nuqtalar polinomning ildizlarini ifodalaydi, qizil kesik chiziq esa birlik doirasini tasvirlaydi. Ildizlarning barchasi birlik doirasida joylashganligini kuzatish mumkin, bu Li-Yang teoremasining holi uchun tasdig'idir.

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