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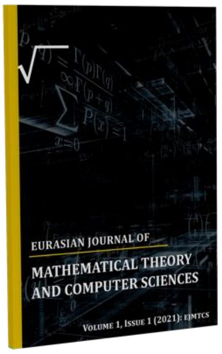

SOLUTION OF THE NEUMANN PROBLEM FOR THE LAPLACE EQUATION IN A RECTANGLE BY THE FOURIER METHOD

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ABSTRACT

The presented article discusses the Fourier method of separation of variables for solving the Laplace equation in a rectangular domain with Neumann boundary conditions. The main stages of this method are subsequently revealed:
1. Representation of the general solution of the Laplace equation in a rectangular domain using Fourier series.
2. Satisfying the Neumann boundary conditions using Fourier series.
3. Constructing the solution of the Neumann problem, including the expansion of the boundary functions into Fourier series, determining the Fourier coefficients, and writing the final solution in the form of a Fourier series.
4. Analysis of the properties of the obtained solution, including its uniqueness, smoothness, continuity, and physical interpretation.

Introduction

The Neumann problem for the Laplace equation is one of the key boundary value problems of mathematical physics. It describes the distribution of physical fields, such as temperature, electric potential, or gravitational potential, within a two- or three-dimensional region for given fluxes or derivatives of these fields at the boundary of the region.

This paper considers the solution of the Neumann problem for the Laplace equation in a rectangular domain by the Fourier variable separation method. This approach allows us to obtain an analytical solution to the problem in the form of a convergent Fourier series.

Solving the Neumann problem in a rectangle is important for many applications, such as:

1. Calculation of temperature fields in the walls and structures of buildings;
2. Modeling the distribution of electric and gravitational fields;
3. Analysis of problems of filtration and diffusion in porous media.

In addition, this problem is a good model problem for studying methods for solving partial differential equations with inhomogeneous boundary conditions. Constructing a solution using Fourier series expansion demonstrates the power of this method and its applicability to a wide range of boundary value problems.

Thus, the study of the solution of the Neumann problem for the Laplace equation in a rectangle by the Fourier method is of both theoretical and practical interest.

Fourier variable separation method

The Neumann problem in a rectangle is given in the Laplace equation, which has some definition. Thus, Laplace's equation is one of the fundamental differential equations in mathematical physics. It has the following definition and properties.

Laplace's equation is a second-order linear partial differential equation that is written in the form:

$$\nabla^2 u = 0$$

Here $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} u(x, y, z)$ is the Laplace operator acting on the scalar field.

Laplace's equation describes stationary (equilibrium) states of various physical fields, such as stationary temperature distribution (thermal conductivity), stationary electric or magnetic fields (electrostatics, magnetostatics), stationary concentration fields (diffusion), stationary hydrodynamic flows (Laplace's equation for the stream function).

Solutions of Laplace's equation have the property of harmonicity - the value of the function at any point is the arithmetic mean of the values at surrounding points. The solutions are infinitely differentiable within the domain of definition where Laplace's equation holds.u

Maximum principle - the maximum and minimum values of the solution are achieved at the boundary of the region, and not inside. To find a unique solution to Laplace's equation, it is necessary to set the appropriate boundary conditions at the boundary of the region.

Dirichlet conditions: $u = f$ at the boundary of the region;

Neumann conditions: $\frac{\partial u}{\partial n} = g$ at the border of the region.

Thus, Laplace's equation is a key equation of mathematical physics, describing a wide range of stationary physical processes. Its properties and boundary conditions determine the uniqueness and smoothness of solutions in various areas.

Now, the Laplace equation in Cartesian coordinates (x, y) for the case under study has the form:

$$\Delta x = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Where is the desired function, representing, for example, temperature, electric or gravitational potential.$u = u(x, y)$

And for the general task:$(x_1, x_2, \dots, x_n)$

$$\Delta x = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

For a rectangular region, the general solution to Laplace's equation can be represented as a Fourier series:$\Omega = \{(x, y) | 0 < x < a, 0 < y < b\}$

$$u(x, y) = \sum_n \left[\begin{array}{c} \left(A_n \cos\left(\frac{n\pi x}{a}\right) + B_n \sin\left(\frac{n\pi x}{a}\right)\right) * \\ * \left(C_n \cosh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi y}{b}\right)\right) \end{array} \right]$$

Where the summation occurs over all natural numbers.n

The Fourier coefficients are determined from the boundary conditions imposed on the function at the edges of the rectangle.$A_n, B_n, C_n, D_n u(x, y)$

This representation of the solution in the form of a double trigonometric series has a number of important properties:

1. Convergence of series in classical function spaces.
2. Ability to satisfy various types of boundary conditions (Dirichlet, Neumann, mixed).
3. Ease of calculating derivatives and integrals of the solution.
4. Visual physical interpretation of the terms of the series.

Thus, the representation of the general solution of the Laplace equation in a rectangle in the form of a Fourier series is a powerful and universal mathematical tool for solving a wide class of boundary value problems.

Satisfying the Neumann boundary conditions using Fourier series when solving the Neumann problem for the Laplace equation in a rectangle by the Fourier method can be done as follows. In the Neumann problem for the Laplace equation in a rectangular region

$$\Omega = \{(x, y) | 0 < x \ < \ a, 0 \ < \ y \ < \ b\}$$

you need to find a function that satisfies:$u(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ в } \Omega$$

$$\frac{\partial u}{\partial n} = g(x, y) \text{ на границе } \Omega$$

Where is a given flow function or derivative on the boundary.$g(x, y)$

To satisfy the Neumann boundary conditions when using the Fourier variable separation method, the solution is represented as a series:$u(x, y)$

$$u(x, y) = \sum_n \left[\begin{array}{c} \left(A_n \cos\left(\frac{n\pi x}{a}\right) + B_n \sin\left(\frac{n\pi x}{a}\right)\right) * \\ * \left(C_n \cosh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi y}{b}\right)\right) \end{array} \right]$$

Fourier coefficients are determined from the conditions:A_n, B_n, C_n, D_n

$$\frac{\partial u}{\partial n} = \sum_n \left[\begin{array}{c} \left(-\frac{n\pi}{a} * A_n \sin\left(\frac{n\pi x}{a}\right) + \frac{n\pi}{a} * B_n \cos\left(\frac{n\pi x}{a}\right)\right) * \\ * \left(C_n \cosh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi y}{b}\right)\right) \end{array} \right] = \ g(x, y)$$

In this case, such an expression is determined at the boundary of the study area. Thus, the boundary value problem is reduced to the problem of determining the coefficients of the Fourier series from a known boundary function. This problem can be solved using various methods, for example:$g(x, y)$

Construction of a solution to the Neumann problem

To construct a solution to the Neumann problem for the Laplace equation in a rectangular domain using the Fourier variable separation method, it is necessary to expand the given boundary function into a Fourier series:$u(x, y) g(x, y)$

$$g(x, y) = \sum_n \left(a_n \cos\left(\frac{n\pi x}{a}\right) + b_n \sin\left(\frac{n\pi x}{a}\right)\right)$$

Where the Fourier coefficients and are determined by the formulas:$a_n b_n$

$$a_n = \left(\frac{2}{a}\right) * \int_0^a g(x, 0) * \cos\left(\frac{n\pi x}{a}\right) dx$$

$$b_n = \left(\frac{2}{a}\right) * \int_0^a g(x,0) * \sin\left(\frac{n\pi x}{a}\right) dx$$

Similarly, it is laid out in a row:$g(x,b)$

$$g(x,b) = \sum_n \left(c_n \cos\left(\frac{n\pi x}{a}\right) + d_n \sin\left(\frac{n\pi x}{a}\right)\right)$$

Where:

$$c_n = \left(\frac{2}{a}\right) * \int_0^a g(x,b) * \cos\left(\frac{n\pi x}{a}\right) dx$$

$$d_n = \left(\frac{2}{a}\right) * \int_0^a g(x,b) * \sin\left(\frac{n\pi x}{a}\right) dx$$

Now, substituting the expansions of boundary functions into the Neumann condition, we obtain:

$$\frac{\partial u}{\partial n} = \left(-\frac{n\pi}{a} * A_n \sin\left(\frac{n\pi x}{a}\right) + \frac{n\pi}{a} * B_n \cos\left(\frac{n\pi x}{a}\right)\right) * \\ * \left(C_n \cosh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi y}{b}\right)\right) = \\ = \sum_n \left(a_n \cos\left(\frac{n\pi x}{a}\right) + b_n \sin\left(\frac{n\pi x}{a}\right)\right) \text{ на } y = 0 \\ = \sum_n \left(c_n \cos\left(\frac{n\pi x}{a}\right) + d_n \sin\left(\frac{n\pi x}{a}\right)\right) \text{ на } y = b$$

From here you can determine the Fourier coefficients by solving a system of linear equations. As a result, we obtain an analytical solution in the form of a Fourier series that satisfies the given Neumann boundary conditions. Thus, the expansion of boundary functions into Fourier series is a key step in constructing a solution to the Neumann problem for the Laplace equation in a rectangle by the method of separation of variables.$A_n, B_n, C_n, D_n, u(x,y)$

To determine the Fourier coefficients, it is necessary to satisfy the Neumann boundary conditions:A_n, B_n, C_n, D_n

$$\frac{\partial u}{\partial n} = \sum_n \left[\begin{matrix} \left(-\frac{n\pi}{a} * A_n \sin\left(\frac{n\pi x}{a}\right) + \frac{n\pi}{a} * B_n \cos\left(\frac{n\pi x}{a}\right)\right) * \\ * \left(C_n \cosh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi y}{b}\right)\right) \end{matrix}\right] = g(x,y)$$

As indicated, the presented expression is determined at the boundary of the region. At the same time, using the expansion of the boundary function g(x, y) into a Fourier series:

$$g(x,0) = \sum_n \left(a_n \cos\left(\frac{n\pi x}{a}\right) + b_n \sin\left(\frac{n\pi x}{a}\right)\right)$$

$$g(x,b) = \sum_n \left(c_n \cos\left(\frac{n\pi x}{a}\right) + d_n \sin\left(\frac{n\pi x}{a}\right)\right)$$

Where the coefficients are determined by the formulas:a_n, b_n, c_n, d_n

$$a_n = \left(\frac{2}{a}\right) * \int_0^a g(x,0) * \cos\left(\frac{n\pi x}{a}\right) dx$$

$$b_n = \left(\frac{2}{a}\right) * \int_0^a g(x,0) * \sin\left(\frac{n\pi x}{a}\right) dx$$

$$c_n = \left(\frac{2}{a}\right) * \int_0^a g(x,b) * \cos\left(\frac{n\pi x}{a}\right) dx$$

$$d_n = \left(\frac{2}{a}\right) * \int_0^a g(x,b) * \sin\left(\frac{n\pi x}{a}\right) dx$$

Substituting these expressions into the Neumann boundary condition, we obtain a system of linear equations for determining the Fourier coefficients:

$$\left(-\frac{n\pi}{a} * A_n\right) * (C_n \cosh(n\pi) + D_n \sinh(n\pi)) = a_n$$
$$\left(\frac{n\pi}{a} * B_n\right) * (C_n \cosh(n\pi) + D_n \sinh(n\pi)) = b_n$$
$$\left(-\frac{n\pi}{a} * A_n\right) * (C_n \cosh(n\pi) - D_n \sinh(n\pi)) = c_n$$
$$\left(\frac{n\pi}{a} * B_n\right) * (C_n \cosh(n\pi) - D_n \sinh(n\pi)) = d_n$$

$a_n, b_n, c_n, d_n$are the Fourier coefficients of the expansion of the boundary $g(x,y)$ function on the boundary of the rectangle.

A_n, B_n– Fourier coefficients in x.

C_n, D_n– Fourier coefficients in y.

Solving this system, we find the coefficients. Substituting them into the expression for , we obtain the final analytical solution of the Neumann problem in the form of a Fourier series.$A_n, B_n, C_n, D_n u(x,y)$

Thus, determining the Fourier coefficients is a key step in constructing a solution to the Neumann problem for the Laplace equation in a rectangle by the method of separation of variables.

The final solution to the Neumann problem for the Laplace equation in a rectangle in the form of a Fourier series has the following form:

$$u(x,y) = \sum_n \left[\begin{array}{c} \left(A_n \cos\left(\frac{n\pi x}{a}\right) + B_n \sin\left(\frac{n\pi x}{a}\right)\right) * \\ * \left(C_n \cosh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi y}{b}\right)\right) \end{array} \right]$$

Where the coefficients are determined from the system of linear equations presented above.A_n, B_n, C_n, D_n

Thus, the solution to the Neumann problem for the Laplace equation in a rectangle is represented as an infinite Fourier series, where each member of the series is the product of trigonometric functions in x and hyperbolic functions in .y

This analytical solution in the form of a Fourier series allows you to obtain detailed information about the behavior of the desired function at any point in the rectangular region, as well as to study the distribution of potential or flux at the boundary.$u(x,y)$

Notes and comments:Solution Properties

The uniqueness of the solution to the Neumann problem is an important property that guarantees that there is one and only one solution for the posed boundary value problem. The existence and uniqueness of a solution to the Neumann problem for the Laplace equation in a bounded domain with a smooth boundary $\partial\Omega$ is proved as follows.

The existence of a solution is proven using variational methods, in particular using the Lax-Milgram theorem; it is shown that the Neumann problem is equivalent to minimizing a linear functional on a set of functions satisfying the Neumann conditions, and from the Lax-Milgram theorem it follows that this minimization problem has a unique solution, which is the solution to the Neumann problem.

The uniqueness of the solution is proved by contradiction. It is assumed that there are two solutions to the Neumann boundary value problem. The difference of these solutions also satisfies the homogeneous Neumann conditions.$w = u_1 - u_2$

It is further shown that on $w \equiv 0$, which is equivalent to Ω, that is, the solution to the Neumann boundary value problem is unique. Thus, the proof of the uniqueness of the solution to the Neumann problem is based on the properties of linear elliptic boundary value problems and the use of variational methods. This is an important result, since it guarantees that the solution to the Neumann problem, found analytically or numerically, will be unique.$u_1 \equiv u_2$

The smoothness and continuity of the solution to the Neumann problem for the Laplace equation in a rectangle has the following properties. Thus, the solution to the Neumann problem for the Laplace equation in a rectangular domain is an infinitely differentiable function inside the domain, that is, smooth. This follows from the fact that Laplace's equation is a linear elliptic equation, the solution of which within the domain has a high degree of smoothness.$u(x,y)\Omega\Omega\nabla^2 u = 0$

Since the Neumann boundary conditions are also smooth at the boundary, the solution will have continuous derivatives of any order inside the rectangle.$\frac{\partial u}{\partial n} = g\partial\Omega u(x,y)\Omega$

The solution to the Neumann problem $u(x,y)$is continuous up to the boundary $\partial\Omega$ of the rectangular region. The proof of the continuity of the solution$\frac{\partial u}{\partial n} = gu(x,y)$ is based on the maximum principle for elliptic equations. Since the Neumann boundary conditions specify a continuous function at the boundary, the solution will also be continuous up to the boundary.

Although the solution is continuous on the boundary $\partial\Omega$, its normal derivative may have discontinuities at the corners of the rectangle. These discontinuities arise due to kinks in the boundary at the corners where the Neumann condition is specified. Despite discontinuities in the derivative, the solution itself remains continuous along the entire boundary. Thus, the solution to the Neumann problem for the Laplace equation in a rectangular region has a high degree of smoothness inside the region and continuity up to the boundary. The features associated with the kinks of the boundary appear only in the behavior of the normal derivative of the solution.

The physical interpretation of the solution to the Neumann problem for the Laplace equation in a rectangle has some definite meaning for the selected results.

1. Simulation in a stationary field of a scalar quantity

The solution to the Neumann problem for the Laplace equation describes the stationary (steady-state) spatial distribution of some scalar quantity in a rectangular region.

This scalar quantity can have a different physical nature: temperature, electric potential, concentration of a substance, etc.

2. Physically specified Neumann boundary conditions

The Neumann boundary conditions at the boundary specify the normal flow of this scalar quantity across the boundary. Physically, this can be interpreted as a given heat flow, electric current, or diffusion flow at the boundary of the region.$\frac{\partial u}{\partial n} = g\partial\Omega$

3. Using internal distribution

The solution inside the rectangular region determines the spatial distribution of the scalar quantity consistent with the specified Neumann boundary conditions. This distribution obeys Laplace's equation, which describes the equilibrium (stationary) state of the system.$u(x, y)\Omega\nabla^2 u = 0$

The solution of the Neumann problem for the Laplace equation in a rectangle finds its practical application in modeling a wide range of physical processes: thermal conductivity, diffusion, electrostatics, hydrodynamics, etc. Knowing the boundary flows, it is possible to determine the internal distribution of the corresponding physical quantity, which is important for the analysis and optimization of these processes.

Thus, solving the Neumann problem for the Laplace equation in a rectangular region has a deep physical meaning and broad practical applications in various fields of science and technology.

Conclusion:

In the course of this work, the solution of the Neumann problem for the Laplace equation in a rectangular region was examined in detail by the Fourier series expansion method.

The main theoretical aspects of the Neumann problem were studied, including proofs of the existence and uniqueness of its solution. It is shown that the Neumann problem is equivalent to minimizing a linear functional, which allows the use of variational methods to find a solution.

A general analytical formula for solving the Neumann problem in the form of an infinite Fourier series is derived. This formula contains Fourier coefficients, which are found from a system of linear equations. Thus, the solution is represented as a sum of products of trigonometric and hyperbolic functions.

The resulting analytical solution in the form of a Fourier series allows us to study in detail the behavior of the desired function u(x,y) in the entire rectangular region, as well as analyze the distribution of potential or flow at the boundary. This is an important property that makes the Fourier method an effective tool for solving the Neumann problem.

In general, the work demonstrates a systematic approach to solving the Neumann problem for the Laplace equation in a rectangle using the Fourier series expansion. The results obtained can be used in analytical research and numerical modeling of a wide class of applied problems described by the Laplace equation with Neumann boundary conditions.

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