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WEAK CONVERGENCE OF STOCHASTIC INTEGRALS OVER POINT PROCESSES IN SPACE *D*

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ABSTRACT

In this paper, we investigate the weak convergence of stochastic integrals to point processes. For clarity, we refer to several accepted assertions from the general theory of random processes, which are detailed in literature sources; therefore, we present formulations without proofs. Here, we utilize concepts from contemporary martingale theory in continuous time, including stochastic calculus in point processes.

Introduction. There are two possible research paths for addressing the problem of weak convergence in measure related to stochastic integrals in point processes.

In the first approach, researchers tend to avoid assuming the continuity of the integrands and instead achieve convergence of the integrals by enhancing the convergence of the processes over which the integration occurs.

In the second approach, no strict requirements are posed on the convergence of the processes, but continuity (in a certain sense) of the integrands is necessary.

We follow the second path and investigate the weak convergence of stochastic integrals over point processes in the Skorokhod topology in space D.

Statement of problem and results. Let *X* be the space of right-continuous piecewise constant functions

$x=\{x_t, t\in R^+\}$ such that $x_0=0$

$x_t<\infty$

$x_t=x_{t^-}+0$ or 1 for all $t<\infty$

Definition. Random process $Z=(Z_t,F_t)$ defined on probability space (Ω,F,P) belonging to space *X* is called a point process (See [1]).

Any point process $Z=(Z_t,F_t)$ is locally bounded and has right-continuous non-decreasing trajectories. Therefore, any point process $Z=(Z_t,F_t)$ has the following decomposition

$Z_t=M_t+A_t$,

where $M=(M_t,F_t),t\in R^+$ is a local martingale with right-continuous trajectories, and $A=(A_t,F_t),t\in R^+$ is a predictable increasing process, called the compensator of the point process $Z=(Z_t,F_t)$.

Consider a sequence of pairs of point processes $(X^n,Y^n)=\left(X_t^n,Y_t^n\right)_{t\geq 0}$, relative to $(F^n,P)X_0^n=Y_0^n=0$ with the following decompositions:

$X_t^n=M_t^n+A_t^n$ and $Y_t^n=N_t^n+B_t^n$.

Consider also a pair of Poisson processes $(X,Y)=(X_t,Y_t)_{t\geq 0}$, $X_0^n=Y_0^n=0$ with compensators $\lambda_1 t$ and $\lambda_2 t$.

Let $f(t,x)$ be a continuous function $(t,R)\in R^+\times R$.

In formulating theorems, we will use the following conditions:

$$A_t^n\xrightarrow{P}\lambda_1 t \qquad \text{(A1)}$$

$$B_t^n\xrightarrow{P}\lambda_2 t \qquad \left\langle M^n,N^n\right\rangle_t\xrightarrow{P}\left\langle M,N\right\rangle_t \qquad \text{(A2)}$$

Theorem 1. Let conditions (A1) and (A2) be satisfied. Then weak convergence in the Skorokhod topology in space *D* holds:

$$\int_0^t f(s,X_{s-}^n)dY_s^n\xrightarrow{D}\int_0^t f(s,X_{s-})dY_s,$$

denoted by V^+ subsets of space *D* consisting of non-decreasing functions.

Definition. Random process $\tau=(\tau_t)_{t\geq 0}$ belonging to class V^+ and such that τ_t is the stopping time (relative to family $F=F(F_t)_{t\geq 0}$) at each $t\geq 0$ is called random time change.

A new process $\hat{X}$ can be related to each process $X\in D\cap F$ and random time change $\tau=(\tau_t)_{t\geq 0}$ assuming that

$$\hat{X}_t(\omega)=X_{\tau_t(\omega)}(\omega),t\geq 0.$$

A new flow of σ - the algebra $\hat{F}=(\hat{F}_t)_{t\geq 0}$ is also related to families F and τ.

Consider $\hat{X}^n=(\hat{X}_t^n,F_t^n)$ and $\hat{Y}^n=(\hat{Y}_t^n,\hat{F}_t^n)$ where $\hat{X}_t^n=\hat{X}_{\tau_t^n}^n$, $\hat{Y}_t^n=\hat{Y}_{\tau_t^n}^n$, $\hat{F}_t^n=\hat{F}_{\tau_t^n}^n$ where $\tau_t^n=(\tau_t^n)_{t\geq 0}\in V^+$.

In formulating theorems, we will use the following conditions:

$$A_{\tau^n(t)}^n\xrightarrow{P}\lambda_1 t \qquad \text{(B1)}$$

$$B_{\tau^n(t)}^n\xrightarrow{P}\lambda_2 t$$

$$\left\langle M^n,N^n\right\rangle_{\tau_t^n}\xrightarrow{P}\left\langle M,N\right\rangle_t \qquad \text{(B2)}$$

Theorem 2. Let conditions (B1) and (B2) be satisfied. Then there is weak convergence in the Skorokhod topology in space *D*

$$\int_0^{\tau_t^n} f(s, X_{s-}^n)dY_s^n \xrightarrow{D} \int_0^t f(s, X_{s-})dY_s$$

The following lemmas are used to prove the theorems.

Lemma 1. Let conditions (A1) and (A2) be satisfied. Then

$$(X^n, Y^n) \xrightarrow{D_f} (X, Y).$$

Lemma 2. Let conditions (B1) and (B2) be satisfied. Then

$$(\hat{X}^n, \hat{Y}^n) \xrightarrow{D_f} (X, Y).$$

Proof of Lemma 1. To prove it, it suffices to show that for $c_1, c_2 \in R$

$$Ee^{ic_1X_t^n + ic_2Y_t^n} \to Ee^{ic_1X_t + ic_2Y_t}$$

we denote

$u_t = e^{ic_1X_t}$ and $z_t = e^{ic_2Y_t}$.

We calculate jumps u_t and z_t:

$$\Delta u_t = u_{t-}(e^{ic_1\Delta X_t} - 1) = u_{t-}(e^{ic_1} - 1)\Delta X_t,$$

$$\Delta z_t = z_{t-}(e^{ic_2\Delta Y_t} - 1) = z_{t-}(e^{ic_2} - 1)\Delta Y_t.$$

Then

$$\Delta u_t \Delta z_t = u_{t-} z_{t-}(e^{ic_1+ic_2} - e^{ic_1} - e^{ic_2} + 1)\Delta X_t \Delta Y_t.$$

By Itô's formula (change of variables)

$$du_t z_t = u_{t-}dz_t + z_{t-}du_t + d[u, z],$$

since

$$du_t = u_{t-}(e^{ic_1} - 1)dX_t$$

$$dz_t = u_{t-}(e^{ic_2} - 1)dY_t$$

Consequently

$$u_t z_z = 1 + \int_0^t u_{s-}z_{s-}(e^{ic_1} - 1)dX_s + \int_0^t u_{s-}z_{s-}(e^{ic_2} - 1)dY_s +$$

$$+\int_0^t u_{s-}z_{s-}(e^{ic_1+ic_2} - e^{ic_1} - e^{ic_2} + 1)d[X, Y].$$

Then

$$Eu_t z_t = 1 + \int_0^t Eu_{s-}z_{s-}(e^{ic_1} - 1)\lambda_1 dt + \int_0^t Eu_{s-}z_{s-}(e^{ic_2} - 1)\lambda_2 dt +$$

$$+\int_0^t Eu_{s-}z_{s-}(e^{ic_1+ic_2} - e^{ic_1} - e^{ic_2} + 1)d\langle X, Y\rangle.$$

Therefore,

$$Ee^{ic_1X_t + ic_2Y_t} = Ee^{(e^{ic_1}-1)\lambda_1 t} \cdot e^{(e^{ic_2}-1)\lambda_2 t} + (e^{ic_1+ic_2} - e^{ic_1} - e^{ic_2} + 1)\langle X, Y\rangle_t.$$

To prove that $(X^n,Y^n)\xrightarrow{D_f}(X,Y)$, it suffices to verify it for each finite time interval. We fix some value of T and denote $(\hat{X}^n,\hat{Y}^n)\xrightarrow{D_f(T)}(X,Y)$ the convergence of finite-dimensional distributions (X^n,Y^n) to finite-dimensional distributions (X,Y) on $[0,T]$.

We determine the sequence (F,P) of stopping times

$\sigma^n, n\geq 1$

$$\sigma^n=\inf\left\{t: A_t^n\geq \lambda_1 T+1\right\}\wedge \inf\left\{t: B_t^n\geq \lambda_2 T+1\right\}.$$

Following (A1), for any $T\geq 0$ as $n\to\infty$ we obtain $P(\sigma^n\leq T)\to 0$.

Let

$$G_t^n(c_1,c_2)=(e^{ic_1}-1)A_t^n+(e^{ic_2}-1)B_t^n+\left(e^{ic_1+ic_2}-e^{ic_1}-e^{ic_2}+1\right)\left\langle M^n,N^n\right\rangle_t.$$

Then from Theorem 5.1.1 ([1])

$$Ee^{ic_1X^n_{t\wedge\sigma^n}+ic_2Y^n_{t\wedge\sigma^n}}\cdot\varepsilon^{-1}_{t\wedge\sigma^n}(G^n)=1$$

and $\left|\varepsilon_{t\wedge\sigma^n}(G^n)\right|\geq d$, where d is a constant independent on $n, \varepsilon_t(G^n)$ - Doléans exponential, i.e.

$$\varepsilon_t(G^n)=e^{G_t^n}\prod_{0\leq s\leq t}(1+\Delta G_s^n)e^{-\Delta G_s^n}$$

then for any $\varepsilon>0$

$$\overline{\lim_n} P\left(\left|\prod_{0\leq s\leq t}(1+\Delta G_s^n)e^{-\Delta G_s^n}-1\right|>\varepsilon\right)=0.$$

From Theorem 5.1.3 [2], it follows that it suffices to check that

$$e^{G_t^n(c_1,c_2)}\xrightarrow{P}e^{G_t(c_1,c_2)}$$

where

$$G_t(c_1,c_2)=(e^{ic_1}-1)\lambda_1 t+(e^{ic_2}-1)\lambda_2 t+(e^{ic_1+ic_2}-e^{ic_1}-e^{ic_2}+1)\left\langle M,N\right\rangle_t.$$

For this, it suffices to show that as $n\to\infty$

$$G_t^n(c_1,c_2)\xrightarrow{P}G_t(c_1,c_2),$$

conditions (A1) and (A2) imply the last expression.

Lemma 2 is proven similar to Lemma 1.

Using Lemma 1, we obtain weak convergence of finite-dimensional distributions

$$\int_o^t f(s,X_{s-}^n)dY_s^n\xrightarrow{Df}\int_0^t f(s,X_{s-})dY_s.$$

Therefore, it suffices to check the relative compactness of the distributions of stochastic integrals

$$\int_o^t f(s,X_{s-}^n)dY_s^n, n\geq 1.$$

Using Theorem 6.4.1 from [2] by conditions (A.1), (A.2), we obtain relative compactnes $\int_o^t f(s, X_{s-}^n) dY_s^n, n \geq 1$.

We define $H_t^n = \int_0^t f(s, M_{s-}^n) dN_s^r$. Let Q^n, $n \geq 1$ be a family of probability distributions on $(D, \Im)$ corresponding to process $H^n, n \geq 1$, i.e., $Q^n(\Gamma) = P\left(H^n \in \Gamma\right)$ for any set $\Gamma \in \Im$. If conditions (A.1) and (A.2) are satisfied, then the family of measures $\left\{Q^n\right\}_{n \geq 1}$ is relatively compact.

To prove the compactness of the family of measures corresponding to process $H^n = \left(H_t^n\right)_{t \geq 0}$, $n \geq 1$, with trajectories in D, it is sufficient to show that:

for any, $T \geq 0$

$$\lim_{a \to \infty} \sup_n P\left(\sup_{0 \leq t \leq T} \left|H_t^n\right| \geq a\right) = 0;$$

for any $T \geq 0$, $\eta > 0$, $\varepsilon_1 > 0$ there exists measure $n_0 = n_0(T, \eta, \varepsilon_1)$ such that for any $\left(F^n, \mathrm{P}\right)$ - stopping times $\tau_n \leq T$, $n \geq n_0$, the following inequality holds:

$$\lim_{\delta \to 0} \sup_{n \geq n_0} P\left(\sup_{0 \leq t \leq \delta} \left|H_{\tau_n + t}^n - H_{\tau_n}^n\right| \geq \eta\right) \leq \varepsilon_1.$$

Consequently

$$\int_o^t f(s, X_{s-}^n) dY_s^n \xrightarrow{D} \int_0^t f(s, X_{s-}^n) dY_s.$$

Relative compactness $\int_o^t f(s, X_{s-}^n) dY_s^n, n \geq 1$ is proven using Theorem 6.4.2 [2].

Consequently, we obtain

$$\int_o^t f(s, X_{s-}^n) dY_s^n \xrightarrow{D} \int_0^t f(s, X_{s-}) dY_s.$$

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