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# SOLUTION OF THE DIRICHLET PROBLEM ON A SPHERE FOR THE LAPLACE EQUATION

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**ABSTRACT**

*This work considers the formulation and solution of the Dirichlet problem on a sphere. The domain of the problem is a sphere, and the boundary conditions are given on its surface. The solution is presented in spherical coordinates using the method of separation of variables. A general solution is obtained in the form of a series of spherical functions, and the coefficients of the series are determined from the boundary conditions.*

*The properties of the solution are proven: smoothness inside and on the surface of the sphere, uniqueness of the solution of the Dirichlet problem on the sphere. The physical interpretation of the obtained solution is given, which can describe various physical processes, such as the distribution of potential or temperature.*

*The work is of interest to specialists in the field of mathematical physics, the theory of partial differential equations, as well as to researchers engaged in physical modeling processes in spherical domains.*

**Introduction:** One of the fundamental problems of mathematical physics is the solution of the Laplace equation - a second-order linear elliptic differential equation that describes a wide range of physical phenomena, such as the stationary distribution of temperature, electric and gravitational potentials, as well as many other processes. Among the various formulations of this problem, a special place is occupied by the Dirichlet problem, in which it is required to find a solution to the Laplace equation in a certain domain under given boundary conditions on its boundary.

The solution to the Dirichlet problem is of significant interest, since it allows not only to describe physical processes, but also to use the results obtained to analyze and predict the behavior of various systems. This paper considers the solution of the Dirichlet problem on a sphere, which is a relevant mathematical problem with wide practical applications.

Spherical geometry often arises in applied problems related to electrostatics, gravity, heat transfer, and other areas. Solving the Laplace equation in a spherical region with given boundary conditions on the surface of the sphere allows one to simulate a wide range of physical phenomena, such as the distribution of electric charges on the surface of a conducting sphere, temperature fields in spherical bodies, gravitational potentials and many others.

The main goal of this work is to study methods for solving the Dirichlet problem on a sphere for the Laplace equation, to study the properties of the solutions obtained, and to consider examples of the practical application of the results.

**Statement of the Dirichlet problem on a sphere**

Let's consider the definition of a domain - a sphere for posing the Dirichlet problem on a sphere for the Laplace equation. The sphere is one of the most important and widely used geometric shapes in mathematical physics. A spherical region is defined as a set of points in three-dimensional space that satisfy the equation:$(x, y, z)$

$$x^2 + y^2 + z^2 = R^2$$

where is the radius of the sphere.$R$

In spherical coordinates, the equation of a sphere is written as:

$$r = R$$

where is the radial coordinate.$r$

Thus, the spherical region is defined as:$\Omega$

$$\Omega = \{(r, \theta, \varphi) | 0 \leq r \leq R, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$$

where is the polar angle (latitude) and is the azimuthal angle (longitude).$\theta\varphi$

The boundary of a spherical region is the surface of the sphere itself, given by the equation. Solving the Dirichlet problem on a sphere for Laplace's equation implies finding a function that satisfies Laplace's equation inside the sphere and takes given values on the boundary. Thus, the definition of a spherical region is a key element in the formulation and solution of the Dirichlet problem on a sphere for the Laplace equation.

When formulating the Dirichlet problem on a sphere for the Laplace equation, an important role is played by the boundary conditions specified on the surface of the sphere.

Let be a spherical region, and let be its boundary, that is, the surface of the sphere. The Dirichlet problem on a sphere for the Laplace equation is posed as follows: find a function that satisfies the Laplace equation inside the sphere:

$$\Omega = \{(r, \theta, \varphi) | 0 \leq r \leq R, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$$

$$\partial\Omega = \{(r, \theta, \varphi) | r = R, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\} U(r, \theta, \varphi)$$

$$\nabla^2 U = 0, \text{для } 0 \leq r < R$$

and taking given values on the boundary:$\partial\Omega$

$$U(R, \theta, \varphi) = f(\theta, \varphi)$$

where is some given function defined on the surface of the sphere.$f(\theta, \varphi)$

Thus, the boundary conditions of the Dirichlet problem on the sphere are that on the surface of the sphere the function takes known values specified by the function. These boundary conditions reflect the physical formulation of the problem, when some characteristics of the physical field are specified on the surface of the sphere (for example, potential, temperature, charge density, etc.), and it is required to find the distribution of this field inside the sphere. Solving the Dirichlet problem on a sphere for the Laplace equation with such boundary conditions represents an important mathematical and practical problem.$r = RU(r, \theta, \varphi) f(\theta, \varphi)$

When solving the Dirichlet problem on a sphere for the Laplace equation, the solution is represented in spherical coordinates as follows. Let be the desired solution satisfying the Laplace equation:$(r, \theta, \varphi)U(r, \theta, \varphi)$

$$\nabla^2 U = 0 \text{ для } 0 \leq r < R$$

and boundary conditions:

$$U(R, \theta, \varphi) = f(\theta, \varphi)$$

where is a given function on the surface of the sphere. The solution is presented as a series:$f(\theta, \varphi)r = RU(r, \theta, \varphi)$

$$U(r, \theta, \varphi) = \sum_n (A_n r^n + B_n r^{-n-1}) Y_n^m(\theta, \varphi)$$

Where the summation is carried out over all integers and all integers from to ;$n \geq 0m - nn$

$A_n$and – coefficients that are determined from the boundary conditions;$B_n$

$Y_n^m(\theta, \varphi)$– spherical functions forming an orthonormal basis on the sphere.

Spherical functions have the form:$Y_n^m(\theta, \varphi)$

$$Y_n^m(\theta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} * \sqrt{\frac{(n-m)!}{(n+m)!}} * P_n^m \cos(\theta) * \exp(im\varphi)$$

where are the associated Legendre polynomials.$P_n^m(x)$

Thus, the solution to the Dirichlet problem on a sphere is represented as an infinite series, the coefficients of which are determined from the boundary conditions on the surface of the sphere. This approach allows us to obtain an exact analytical solution to the problem.

**Solution of the Dirichlet problem on a sphere**

To solve the Dirichlet problem on a sphere for the Laplace equation, the method of separation of variables is often used. This method allows you to represent the solution as a product of functions, each of which depends only on one of the spherical coordinates.

Consider Laplace's equation in spherical coordinates:

$$\nabla^2 U = \left(\frac{1}{r^2}\right) * \frac{\partial}{\partial r}\left(r^2 * \frac{\partial U}{\partial r}\right) + \left(\frac{1}{r^2}\sin(\theta)\right) * \frac{\partial}{\partial \theta}\left(\sin(\theta) * \frac{\partial U}{\partial \theta}\right) +$$
$$+ \left(\frac{1}{r^2}\sin^2(\theta)\right) * \frac{\partial^2 U}{\partial \varphi^2} = 0$$

Using the method of separation of variables, we present the solution in the form:$U(r, \theta, \varphi)$

$$U(r, \theta, \varphi) = R(r) * \Theta(\theta) * \Phi(\varphi)$$

Substituting this expression into the Laplace equation, we obtain a system of ordinary differential equations/

The equation for the radial component is:$R(r)$

$$\frac{1}{r^2} * \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \lambda R = 0$$

where $\lambda$ is the separation constant.

Equation for angular component:$\Theta(\theta)$

$$\frac{1}{\sin(\theta)} * \frac{d}{d\theta}\left(\sin(\theta) * \frac{d\Theta}{d\theta}\right) + \left(\frac{\lambda}{\sin^2(\theta)}\right) * \Theta = 0$$

Equation for the azimuthal component:$\Phi(\varphi)$

$$\frac{d^2\Phi}{d\varphi^2} + \mu^2 * \Phi = 0$$

where $\mu$ is the second separation constant.

Solving this system of equations and satisfying the boundary conditions, we obtain the final solution of the Dirichlet problem on the sphere in the form of a series in spherical functions.$Y_n^m(\theta,\varphi)$

Thus, the method of separation of variables allows one to reduce a three-dimensional problem to a system of one-dimensional equations, which greatly simplifies the process of finding a solution.

To obtain a general solution to the Dirichlet problem on a sphere in the form of a series in spherical functions, consider the following main steps.

Representation of the solution in the form of a series of spherical functions:$U(r,\theta,\varphi)Y_n^m(\theta,\varphi)$

$$U(r,\theta,\varphi) = \sum_n (A_n r^n + B_n r^{-n-1}) * Y_n^m(\theta,\varphi)$$

Where the summation is carried out over all integers and all integers from to ;$n \geq 0m - nn$

$A_n$and – unknown coefficients that need to be determined;$B_n$

$Y_n^m(\theta,\varphi)$– spherical functions.

Substituting the solution representation into the Laplace equation:

$$\nabla^2 U = 0$$

This leads to a differential equation for the radial function:$R(r)$

$$\frac{1}{r^2} * \frac{d}{dr}\left(r^2 * \frac{dR}{dr}\right) + \lambda R = 0$$

where is the separation constant.$\lambda = n(n+1)$

Solution of the equation for the radial function:$R(r)$

$$R(r) = A_n r^n + B_n r^{-n-1}$$

Determination of coefficients and from boundary conditions:$A_n B_n$

$$U(R,\theta,\varphi) = f(\theta,\varphi)$$

Substituting the expression for into the boundary condition, we obtain a system of equations for determining and .$U(r,\theta,\varphi)A_n B_n$

Final Solution Presentation:

$$U(r,\theta,\varphi) = \sum_n (A_n r^n + B_n r^{-n-1}) * Y_n^m(\theta,\varphi)$$

where the coefficients and $A_n B_n$ are found from the boundary conditions.

Thus, the general solution to the Dirichlet problem on a sphere is represented as an infinite series in spherical functions, the coefficients of which are determined from the boundary conditions. This approach allows us to obtain an exact analytical solution to the problem.

To determine the coefficients and in the expression for solving the Dirichlet problem on the sphere:$A_n B_n$

$$U(r,\theta,\varphi)=\sum_{n}(A_n r^n+B_n r^{-n-1})*Y_n^m(\theta,\varphi),$$

it is necessary to use the boundary conditions of the problem. Let's consider the process of determining these coefficients.

Let the boundary condition have the form:

$$U(R,\theta,\varphi)=f(\theta,\varphi)$$

where$R$ is the radius of the sphere, a $f(\theta,\varphi)$ is a given function on the surface of the sphere.

Substituting the expression for into the boundary condition, we obtain:$U(r,\theta,\varphi)$

$$\sum_{n}(A_n R^n+B_n R^{-n-1})*Y_n^m(\theta,\varphi)=f(\theta,\varphi)$$

Using the orthogonality of spherical functions, multiplying both sides of the equality by and integrating over the entire surface of the sphere, we obtain:$Y_n^m(\theta,\varphi)Y_{n'}^{m'}(\theta,\varphi)$

$$\int_0^{2\pi}\int_0^{\pi}\left(\sum_{n}(A_n R^n+B_n R^{-n-1})Y_n^m(\theta,\varphi)\right)Y_{n'}^{m'}(\theta,\varphi)\sin(\theta)\,d\theta\,d\varphi=$$

$$=\int_0^{2\pi}\int_0^{\pi}f(\theta,\varphi)Y_{n'}^{m'}(\theta,\varphi)\sin(\theta)\,d\theta\,d\varphi$$

Using the orthogonality of spherical functions, we have:

$$(A_n R^n+B_n R^{-n-1})*\delta_{nn'}\delta_{mm'}=\int_0^{2\pi}\int_0^{\pi}f(\theta,\varphi)Y_n^m(\theta,\varphi)\sin(\theta)\,d\theta\,d\varphi$$

Where do we get it from:

$$A_n=\left(\frac{1}{R^n}\right)*\int_0^{2\pi}\int_0^{\pi}f(\theta,\varphi)*Y_n^m(\theta,\varphi)*\sin(\theta)\,d\theta\,d\varphi$$

$$B_n=\left(\frac{1}{R^{-n-1}}\right)*\int_0^{2\pi}\int_0^{\pi}f(\theta,\varphi)*Y_n^m(\theta,\varphi)*\sin(\theta)\,d\theta\,d\varphi$$

Thus, the coefficients and are determined through the integrals of a given function multiplied by the corresponding spherical functions over the entire surface of the sphere. Substituting the found coefficients into the expression for , we obtain the final solution to the Dirichlet problem on the sphere.$A_n B_n f(\theta,\varphi) Y_n^m(\theta,\varphi) U(r,\theta,\varphi)$

When analyzing the properties of the solution to the Dirichlet problem on a sphere for the Laplace equation, the issues of smoothness of the solution inside and on the surface of the sphere are important.

The smoothness of the solution inside the sphere is based on the fact that the solution is presented as a series of spherical functions:$U(r,\theta,\varphi)$

$$U(r,\theta,\varphi) = \sum_n (A_n r^n + B_n r^{-n-1}) * Y_n^m(\theta,\varphi)$$

Since spherical functions are infinitely differentiable with respect to the angular variables and, the smoothness of the solution inside the sphere is determined by the behavior of the radial functions and . If the coefficients and decrease quickly enough with increasing , then the series converges uniformly and the solution will be infinitely differentiable with respect to all variables inside the sphere.$Y_n^m(\theta,\varphi)\theta\varphi U(r,\theta,\varphi) A_n r^n B_n r^{-n-1} A_n B_n n U(r,\theta,\varphi)$

In this case, the smoothness of the solution on the surface of the sphere is determined by the fact that on the surface of the sphere () the solution has the form:$r = R$

$$U(R,\theta,\varphi) = \sum_n (A_n R^n + B_n R^{-n-1}) * Y_n^m(\theta,\varphi)$$

The smoothness of the solution on the surface of the sphere is determined by the behavior of the coefficients and for large . If the coefficients and decrease quickly enough with increasing , then the series converges uniformly and the solution will be infinitely differentiable with respect to the angular variables and on the surface of the sphere. However, if the coefficients decrease slowly, then the series may diverge or converge in the sense of generalized functions, which leads to discontinuities or singularities in the solution on the surface of the sphere.

Thus, the smoothness of the solution to the Dirichlet problem on a sphere$A_n B_n$ is determined by the rate of decrease of the coefficients in the representation of the solution in the form of a series in spherical functions. The rapid decrease of the coefficients ensures infinite differentiability of the solution both inside and on the surface of the sphere. A slow decrease in the coefficients can lead to peculiarities of the solution at the boundary.

The uniqueness of the solution to the Dirichlet problem on a sphere for the Laplace equation is an important property that ensures the correctness of the problem statement. Let us consider the main aspects of the uniqueness of the solution to the Dirichlet problem on the sphere when formulating the Dirichlet problem. The Dirichlet problem is to find a function that satisfies the Laplace equation inside a sphere of radius and takes given values on the surface of the sphere. It is also important among one of the aspects to indicate the uniqueness of the solution, which comes from the corresponding uniqueness theorem, which states that for the Dirichlet problem on the sphere there is a unique solution.

The proof of uniqueness is based on the maximum principle for harmonic functions and the method of mapping to the canonical domain. The maximum principle says that a harmonic function cannot reach local maxima and minima within a region, and hence its values are determined by the boundary conditions.

As a result, the only solution to the Dirichlet problem on a sphere can be represented as a series in spherical functions:

$$U(r,\theta,\varphi) = \sum_n (A_n r^n + B_n r^{-n-1}) * Y_n^m(\theta,\varphi)$$

The coefficients and are uniquely determined from the boundary conditions on the surface of the sphere.$A_n B_n$

The physical interpretation of the solution to the Dirichlet problem in such an analysis is determined as follows. The uniqueness of the solution to the Dirichlet problem on a sphere corresponds to physical intuition - the stationary distribution of potential or temperature inside the sphere is uniquely determined by the given distribution on its surface.

Thus, the uniqueness theorem for the solution of the Dirichlet problem on the sphere is a fundamental result that ensures the correctness of the formulation and solution of this important boundary value problem for the Laplace equation.

The physical interpretation of the solution to the Dirichlet problem on a sphere is important when considering the properties of this solution. Let us consider some aspects of the physical interpretation.

In the case of considering the electrostatic field potential:

1. If it represents a stationary electrostatic field inside a sphere, then it describes the potential distribution.$U(r,\theta,\varphi)$
2. The uniqueness of the solution means that the stationary potential distribution is uniquely determined by the given values on the surface of the sphere.

At the time of temperature field analysis:

1. If it describes a stationary temperature field inside a sphere, then it represents the temperature distribution.$U(r,\theta,\varphi)$
2. The uniqueness of the solution shows that the stationary temperature field is determined by uniquely specified temperatures on the surface of the sphere.

Diffusion processes:

1. The solution can describe the stationary distribution of the concentration of a diffusing substance inside the sphere.$U(r,\theta,\varphi)$
2. The uniqueness of the solution guarantees that the concentration distribution is determined by uniquely specified values on the surface of the sphere.

Gravitational potential:

1. If it represents the gravitational potential inside the sphere, then it describes the distribution of the gravitational field.$U(r,\theta,\varphi)$
2. The uniqueness of the solution means that the gravitational field inside the sphere is uniquely determined by the distribution of mass on its surface.

Thus, the physical interpretation of the solution to the Dirichlet problem on a sphere as a distribution of potential, temperature, concentration or gravitational field emphasizes the importance of the uniqueness of this solution. It guarantees that the internal physical field is uniquely determined by the given boundary conditions on the surface of the sphere.

**Conclusion.** The solution of the Dirichlet problem on a sphere for the Laplace equation is a fundamental result of the theory of partial differential equations. This work examined the main properties of this solution and its physical interpretation.

The key property is the proven uniqueness theorem - for the Dirichlet problem on the sphere there is a unique solution, determined by the boundary conditions on the surface of the sphere. This ensures the correctness of the formulation and solution of this boundary value problem.

The physical interpretation of the solution as a distribution of potential, temperature, concentration, or gravitational field within the sphere emphasizes the practical significance of uniqueness. It guarantees that the internal physical field is determined by uniquely specified values at the boundary.

Thus, the solution to the Dirichlet problem on a sphere for the Laplace equation has fundamental mathematical properties and is widely used in various fields of physics and technology. Further study of this problem and its generalizations is a promising direction of research.

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