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SEPARATE SCHEMES FOR SPECIFIC PRODUCTIVE DIFFERENTIAL EQUATIONS

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ABSTRACT

This article is devoted to the development of differential schemes for special derivative differential equations, in which differential schemes for solving differential differential equations are suitable differential templates, differential and differential problems for diffusion, approximation of differential schemes, constant flour and convergence, an algorithm for solving the problem of thermal conductivity with a weighted scheme is given.

XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR UCHUN AYIRMALI SXEMALAR

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KALIT SO'ZLAR

Ayirmali sxemalar, diffuziya masalalari, xususiy hosilali differensial tenglamalar, approksimatsiya, turg'unlik.

ANNOTATSIYA

Ushbu maqola xususiy hosilali differensial tenglamalar uchun ayirmali sxemalar tuzishga bag'ishlangan bo'lib, unda xususiy hosilali differensial tenglamalarni yechish uchun ayirmali sxemalar ularga mos ayirmali shablonlar, diffuziya masalasi uchun differensial va ayirmali masalalarni qo'yish, ayirmali sxemalarning approksimatsiyasi, turg'unligi va yaqinlashishi, issiqlik o'tkazuvchanlik masalasini vaznli sxema bilan yechish algoritmi keltirilgan.

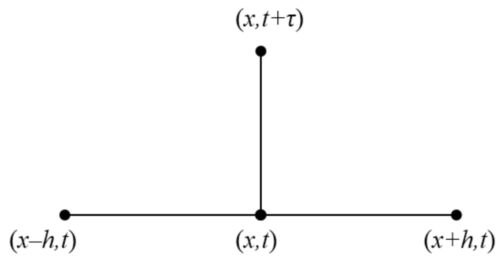
Ko'pgina ayirmali sxemalarni tuzishda differensial chegaraviy masalaning yetarlicha silliq yechimi mavjud deb faraz qilinadi hamda differensial tenglamadagi hosilalar taqriban ayirmali munosabatlarga almashtiriladi. Ammo ko'pgina muhim fizik jarayonlarni tavsiflashda funksiyalarning differensiallanuvchi bo'lishi yetarli emas. Masalan, fizik eksperimentlar ko'rsatadiki, bosim, zichlik va temperaturaning qovushqoq bo'lmagan, tovushdan tez gaz oqimlardagi taqsimotini tavsiflashda shunday funksiyalar paydo bo'ladiki, ular sakrash – zarbali to'lqinlar bilan xarakterlanadi. Sakrashlar boshlang‘ich shartlar silliq bo‘lganda ham vaqt og‘ishi bilan paydo bo‘lishi mumkin.

Xususiy xosilali differensial tenglamalarni yechish uchun ayirmali sxemalar

a) $$L\upsilon = \frac{\partial \upsilon}{\partial t} - \frac{\partial^2 \upsilon}{\partial x^2}, \upsilon = \upsilon(x,t)$$

xususiy xosilali differensial operatorlarni qaraylik.

Tekislik (x,t) da x va t nuqtalarni belgilab olaylik $h>0$ va $\tau>0$ – ikkita son (to`r qadamlari) bo`lsin. Differensial operator L ning ayirmali approksimatsiyasi $L_{h\tau}$ ni yozish uchun, avvalambor tegishli shablonni aniqlash lozim. Shablon bu ayirmali sxema aniqlangan to`r tugunlari to`plamidan iborat. Ayirmali shablon 1a-rasmda keltirilgan to`rt nuqtali shablonda aniqlangan bo`lsin:



1a-rasm

Bu holda ayirmali operator $L_{h\tau}$ ni quyidagicha aniqlaymiz

$$L_{h\tau}^{(0)}\upsilon = \frac{\upsilon(x,t+\tau)-\upsilon(x,t)}{\tau} - \frac{\upsilon(x+h,t)-2\upsilon(x,t)-\upsilon(x-h,t)}{h^2} \quad (1)$$

Ayirmali sxemalarning yozilishini soddalashtirish maqsadida quyidagi shartli belgilardan foydalaniladi:

$$\upsilon = \upsilon(x,t) \quad \hat{\upsilon} = \upsilon(x,t+\tau),$$

$$\check{\upsilon} = \upsilon(x,t-\tau).$$

Kiritilgan belgilashlar asosida, masalan t bo`yicha ayirmali hosila quyidagicha yozilishi mumkin

$$\upsilon_t = \frac{\upsilon(x,t+\tau)-\upsilon(x,t)}{\tau} = \frac{\hat{\upsilon}-\upsilon}{\tau}. \quad (2)$$

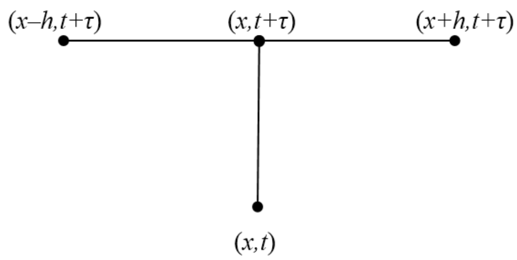
Bu holda formula (6.1) ni ushbu ko`rinishda yozish mumkin

$$L_{h\tau}^{(0)}\upsilon = \upsilon_t - \upsilon_{\bar{x}x}, \quad (3)$$

bunda

$$\vartheta_{\bar{x}x} = \frac{\vartheta(x+h,t) - 2\vartheta(x,t) + \vartheta(x-h,t)}{h^2}$$

ayirmali operator $L_{h\tau}^{(0)}$ ni tuzishda biz $\vartheta_{\bar{x}x}$ ning qiymatini vaqt bo`yicha quyi qatlamda oldik. 1b-rasmda tasvirlangan shablondan foydalansak



1b-rasm

$\vartheta_{\bar{x}x}$ ni vaqtning $t+\tau$ (yuqori qatlamda) momentida olishimiz mumkin, bu holda

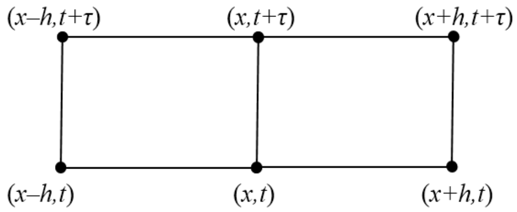
$$L_{h\tau}^{(1)}\upsilon = \upsilon_t - \hat{\upsilon}_{\bar{x}x} \quad (4)$$

ayirmali operator o’rinli bo’ladi.

Endi (6.3) va (6.4) ning chiziqli kombinatsiyasini qarasak, ayirmali sxemalarning bir parametrli oilasi

$$L_{h\tau}^{(\sigma)}\vartheta = \vartheta_t - (\sigma\hat{\vartheta}_{\bar{x}x} + (1-\sigma)\vartheta_{\bar{x}x} \quad (5)$$

ni hosil qilamiz, ushbu sxema $\sigma \neq 0$ va $\sigma \neq 1$ bo`lganda olti nuqtali shablonda aniqlangan, bu shablon 1c-rasmda tasvirlangan



1c-rasm

$L\upsilon = \frac{\partial^2 \upsilon}{\partial t^2} - \frac{\partial^2 \upsilon}{\partial x^2}$ - differensial operatorni qaraylik. Operatorda vaqt bo`yicha ikkinchi tartibli hosila mavjudligi sababli to`r uch qatlamli bo`lib to`r funksiyasi vaqtning $t-\tau, t, t+\tau$ momentlarida qaraladi. Bunda minimal shablon besh nuqtali shablon bo`ladi (rasm 2,a,b,c).

Ushbu shablonlardagi mumkin bo`lgan approksimatsiyalardan biri (shablon 2c rasm) da $\vartheta_{\bar{x}x}$ ning qiymati vaqt $t$, ning o`rta qatlamida topiladi

$$L_{h\tau} = \vartheta_{\bar{t}t} - \hat{\vartheta}_{\bar{x}x} \quad (6)$$

bu yerda

$$\vartheta_{\bar{t}t}(x,t) = \frac{\vartheta(x,t+\tau) - 2\vartheta(x,t) + \vartheta(x,t-\tau)}{\tau^2}$$

.

Xuddi shu tariqa (shablon 2a) da aniqlangan operatorni yozish mumkin.

$$L_{h\tau} = \vartheta_{\bar{t}t} - \hat{\vartheta}_{\bar{x}x}.$$

To`qqiz nuqtali shablon(2d rasm) da ikki parametrli ayirmali sxemalar oilasini qarash mumkin

$$L_{h\tau}^{(\delta_1,\delta_2)}\vartheta = \vartheta_{\bar{t}t} - (\sigma_1\vartheta_{\bar{x}x} + (1-\sigma_1-\sigma_2)\vartheta_{\bar{t}t} + \sigma_2\vartheta_{\bar{t}t})$$

.

c) Ushbu differensial operator $L\upsilon = \upsilon''$ ning noregulyar shablon (tengmas oraliqli to`rdagi) approksimatsiyasini qaraymiz.

$h_- > 0$ va $h_+ > 0$ ikkita ixtiyoriy son bo`lsin. Uch nuqtali shablonni olamiz. Agarda $h_- \neq h_+$ , bo`lsa shablon noregulyar deyiladi (bunday shablonlardan tashkil topgan to`r tengmas oraliqli deyiladi). Quyidagi belgilashni kiritib

$$\upsilon_{\bar{x}} = \frac{\upsilon(x) - \upsilon(x-h_-)}{h_-},$$

$$\upsilon_x = \frac{\upsilon(x+h_+) - \upsilon(x)}{h_+}, \quad \hbar = 0,5(h_- + h_+)$$

ayirmali operator $L_h\upsilon$ ni ushbu formula bo`yicha aniqlaymiz

$$L_h\upsilon = \frac{1}{\hbar}\left[\frac{\upsilon(x+h_+) - \upsilon(x)}{h_+} - \frac{\upsilon(x) - \upsilon(x-h_-)}{h_-}\right] = \frac{\upsilon_x - \upsilon_{\bar{x}}}{\hbar} \quad (7)$$

Ayirmali sxemaning turg'unligiga ham to`xtalib o'tamiz. Odatda boshlang'ich shart bo`yicha o`ng tomondagi funksiya bo`yicha turg'unlik tushunchalaridan foydalaniladi.

1-*Ta'rif.* Ayirmali sxema boshlang'ich shart bo`yicha turg'un deyiladi, agar birjinsli tenglamaning

$$By_t + Ay = 0, \quad t = n\tau > 0, \quad y(0) = y_0, \qquad (8)$$

yechimi uchun ushbu baho o`rinli bo`lsa

$$\| y_{h\tau}(t+\tau) \|_{(1h)} \leq M_1 \| y_{0h} \|_{(1h)} \quad (9)$$

2-*Ta'rif.* Ayirmali sxema o`ng tomon bo`yicha turg'un deyiladi, agarda tenglamaning birjinsli boshlang'ich shart $y(0) = 0$ bilan olingan yechimi uchun, ya'ni

$$By_t + Ay = \varphi, y(0) = 0, \qquad (10)$$

tenglamaning yechimi uchun quyidagi baho o`rinli bo`lsa

$$\| y_{h\tau}(t+\tau) \|_{(1h)} \leq M \max_{0 \leq t' \leq t} \| \varphi_{h\tau}(t') \|_{(2h)}. \quad (11)$$

Ayirmali masala ning yechimini $y = y^{(1)} + y^{(2)}$ ko`rinishida ifodalaymiz, bu yerda $y^{(1)}$-masala (8)(ning yechimi, $y^{(2)}$ esa masala (9) ning yechimi.

Uchburchaklar tengsizligiga asosan

$$\| y_{h\tau}(t+\tau) \|_{(1h)} \le \| y_{h\tau}^{(1)}(t+\tau) \|_{(1h)} + \| y_{h\tau}^{(2)}(t+\tau) \|_{(1h)}$$

tengsizlikka, yoki tengsizliklar (10) va (11) ni e'tiborga olgan holda

$$\| y_{h\tau}(t+\tau) \|_{(1h)} \le M_1 \| y_{h\tau}^{(1)}(t+\tau) \|_{(1h)} + M_2 \max_{0 \le t' \le t} \| \varphi_{h\tau}(t') \|_{(2h)} \quad (12)$$

munosabatga ega bo`lamiz. Ushbu tengsizlik Koshi masalasi $By_t + Ay = \varphi(t), t = t_n = n\tau \in w_\tau, y(0) = y_0 \in B_h$, ning boshlang'ich shartlardan $h$ va $\tau$ bo`yicha tekis va uzliksiz bog'liq ekanligini anglatadi. Ana shu shart turg'unlik deb ataladi.

3-*Ta'rif.* Ayirmali sxema $By_t + Ay = \varphi(t), t = t_n = n\tau \in w_\tau, y(0) = y_0 \in B_h$, ni absolyut turg'un deyiladi, agarda u ixtiyoriy h va τ lar uchun turg'un bo`lsa (nafaqat yetarlicha kichik).

Shart 12 ni qisqacha quyidagicha yozish kerak:

$$\| y\ (t+\tau) \|_{(1)} \le M_1 \| y_0 \|_{(1)} + M_2 \max_{0 \le t' \le t} \| \varphi_{h\tau}(t') \|_{(2)}$$

Ayirmali sxemalarning turg'unligini tadqiq etishda $\| * \|_{(1)}$ sifatida asosiy norma $\| y \| = \sqrt{(y,y)}$ bilan bir qatorda energetik normalardan

$$\| y \|_A = \sqrt{(Ay,y)},\ \text{agar}\ A = A^x > 0\ \text{da},$$

$$\| y \|_B = \sqrt{(By,y)},\ \text{agar}\ B = B^x > 0\ \text{da}$$

foydalanamiz. Ayirmali sxemalarning turg'unligi ayirmali sxemalarning biror-bir dastlabki oilasida olib boriladi. Dastlabki sxemalar oilasi quyidagicha aniqlanadi:

1) A va B operatorlar t dan bog'liq emas (o`zgarmas operatorlar),

2) B operator musbat, $B > 0$

3) A-o`z-o`ziga qo`shma va musbat operator, $A = A^* > 0$.

Shartlarni 1)-3) va yechimga ega bo`lish sharti barcha mumkin bo`lgan sxemalardan to`plamidan yo`l qo`yilishi mumkin bo`lgan (dastlabki oila) sxemalar oilasini ajratib olishga imkon beradi.

Ayirmali sxema $By_t + Ay = \varphi(t), t = t_n = n\tau \in w_\tau, y(0) = y_0 \in B_h$, ning turg'unligini energetik tengsizliklar metodi bilan olib beramiz. Tenglama $By_t + Ay = \varphi(t), t = t_n = n\tau \in w_\tau, y(0) = y_0 \in B_h$, ni skalyar ravishda $2\tau y_t = 2(\hat{y} - y)$ ga ko`paytiramiz:

$$2\tau(By_t, y_t) + 2\tau(Ay, y_t) = 2\tau(\varphi, y_t) \quad (13)$$

Ushbu formuladan foydalangan holda

$$y = \frac{\hat{y} + y}{2} - \frac{\hat{y} - y}{2} = \frac{1}{2}(\hat{y} + y) - \frac{\tau}{2} y_t, \quad (14)$$

formula (6.31) ni quyidagicha yozib olamiz

$$2\tau((B - 0{,}5EA)y_t, y_t) + (A(\hat{y} + y), \hat{y} - y) = 2\tau(\varphi, y_t). \quad (15)$$

1-*Lemma.* A o`z o`ziga qo`shma operator bo`lsin. U holda ushbu munosabat o`rinli bo`ladi

$$(A(\hat{y} + y), \hat{y} - y) = A(\hat{y}, \hat{y}) - A(y, y). \quad (16)$$

Isbot. Haqiqatan ham

$$(A(\hat{y} + y), \hat{y} - y) = (A\hat{y}, \hat{y}) + (Ay, \hat{y}) - (A\hat{y}, y) - (Ay, y) = (\hat{y}, \hat{y}) - A(y, y).$$

o`rinli bo`ladi, chunki A operatorning o`z-o`ziga qo`shmaligiga asosan

$$(Ay,\hat{y})=(y,A\hat{y})=(A\hat{y},y).$$

tenglik o`rinli.

Endi (16) ni (14) ga qo`yib, ayirmali sxema $By_t+Ay=\varphi(t), t=t_n=n\tau\in w_\tau, y(0)=y_0\in B_h$, uchun energetik ayniyat hosil qilamiz.

$$2\tau((B-0,5\tau A)y_t,y_t)+(A\hat{y},\hat{y})=(Ay,y)+2\tau(\varphi,y_t). \quad (17)$$

Ikki qatlamli sxema (8) ning H_A fazoda boshlang'ich shart bo`yicha turg'unligini qaraymiz.

1-*Teorema.* Dastlsabki sxemalar oilasiga mansub bo`lgan sxema (8) ning $H_A$ fazoda boshlang'ich shart bo`yicha $M_1=1$ o`zgarmas bilan turg'un bo`lishi, ya'ni masala (8) ning yechimi y_n uchun

$$\| y_N \|_A \le \| y_0 \|_A, n=1,2,..., \quad (18)$$

baho o`rinli bo`lishi uchun

$$B\ge\frac{\tau}{2}A \qquad (19)$$

shartning bajarilishi zarur va yetarlidir.

***Isbot.* Yetarlilik.** Shart (19) bajarilgan bo`lsin, masala uchun $(\varphi=0)$ energetik ayniyat (16) dan quyidagi ayniyatga ega bo'lamiz

$$2\tau((B-0,5\tau A)y_t,y_t)+(A\hat{y},\hat{y})=(Ay,y). \quad (20)$$

bunda ushbu tengsizlik $(A\hat{y},\hat{y})\le(Ay,y)$. yoki $\| \hat{y}_N \|^2_A \le \| y \|^2_A$, kelib chiqadi, bundan o`z navbatida

$$\| y_{n+1} \|_A \le \| y_n \|_A \le ... \le \| y_0 \|_A$$

ekanligi yoki tengsizlik (19) o`rinli ekanligi kelib chiqadi.

Zaruriylik. Ayirmali sxema (8) turg'un va tengsizlik (18) bajarilgan bo`lsin. Bundan operatorli tengsizlik (19), ya'ni

$$(B\upsilon,\upsilon)\ge 0,5\tau(A\upsilon,\upsilon) \text{ ixtiyoriy } \upsilon\in H \quad (20)$$

uchun bajarilishligini isbotlaymiz.

Ayniyat (20) dan birinchi qatlam uchun $(n=0)$ quyidagiga ega bo`lamiz:

$$2\tau((B-0,5\tau A)y_t(0),y_t(0))+(Ay_1,y_1)=(Ay_0,y_0).$$

Tengsizlik (18) o`rinli bo`lganligi uchun ushbu tengsizlik faqat

$$2\tau((B-0,5\tau A)y_t(0),y_t(0))=(Ay_0,y_0)-(Ay_1,y_1)\ge 0$$

bo`lgandagina bajariladi, ya'ni

$$2\tau((B-0,5\tau A)y_t(0),y_t(0))\ge 0 \quad (21)$$

bo`ladi. Chunki $y_0\in H$ ixtiyoriy element, u holda $\upsilon=y_t(0)=-B^{-1}Ay_0\in H$ ham ixtiyoriy bo`ladi.

Haqiqatan ham, ixtiyoriy $\upsilon=y_t(0)\in H$ elementni berib, $y_0=-A^{-1}By_0\in H$ ekanligini topamiz, chunki A^{-1} operator mavjud. Shunday qilib, tengsizlik (21) ixtiyoriy $\upsilon=y_t(0)\in H$ uchun o`rinli, ya'ni operatorli tengsizlik (19) o`rinli, talab qilingan mulohaza isbotlanadi.

Turg'unlik sharti (19) ning samaradorligini vaznli sxema misolida ko`rib o`tamiz

$$y_t+A(\sigma\hat{y}+(1-\sigma)y)=0.$$

Ushbu sxemaning kanonik ko`rinishi

$$(E+\sigma\tau A)y_t + hy = 0,\;\; B = E+\sigma\tau A \qquad (22)$$

ekanligi ma'lum. Agarda $A = A^* > 0$ va u t dan bog'liq bo`lmasa, hamda $\delta > -\frac{1}{\tau \| A \|}$ bo`lsa, u holda vaznli sxema (22) dastlabki sxemalar oilasiga mansub bo`ladi. Ayirmali sxema turg'unligining zaruriylik va yetarlilik sharti (19) quyidagi ko`rinishni oladi

$$B - 0.5\tau A = E + (\sigma - 0.5)\tau A \geq 0\,.$$

Endi $A \leq \|A\| E$ va $E \geq A / \|A\|$ ekanligini e'tiborga olib ushbuga ega bo`lamiz

$$B - 0.5\tau A \geq \left(\frac{1}{\tau \|A\|} + (\sigma - 0.5)\tau\right) A$$

Bundan ko`rinadiki, shart (19) ushbu tengsizlikka ekvivalent

$$\sigma \geq \sigma_0, \text{ bu yerda } \sigma_0 = \frac{1}{2} - \frac{1}{\tau \|A\|}.$$

Ana shu shart vaznli sxemaning turg'unligi uchun zarur va yetarlidir.

Ayirmali masala differensial masalani approksimatsiyalasa va turg'un bo`lsa, albatta u yaqinlashuvchi bo`ladi.

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