PHASES OF THE FERROMAGNETIC POTTS MODEL WITH FIVE STATES ON A FIRST-ORDER BETHE LATTICE

G'anixo'jayev Nosir
Tursunqulov Abduazim
Samiyeva Fayyoza
M.Sc. degree in the Department of Theoretical Physics, National University of Uzbekistan,
Tashkent, Republic of Uzbekistan.
https://doi.org/10.5281/zenodo.13885351

ARTICLE INFO

Qabul qilindi: 25- Sentabr 2024 yil
Ma'qullandi: 28- Sentabr 2024 yil
Nashr qilindi: 30- Sentabr 2024 yil

KEY WORDS

Potts model, Ising model, first-order Bethe lattice, phase transitions.

ABSTRACT

The Ferromagnetic Potts model with five states on a first-order Bethe lattice is a significant subject in statistical physics and mathematical modeling. This abstract explores the distinct phases exhibited by this model and their implications. The Potts model is a lattice-based statistical mechanics model that generalizes the Ising model to systems with more than two possible states at each lattice site. In this study, we focus on the ferromagnetic variant of the Potts model with five states, which is particularly interesting due to its richer phase behavior. Utilizing the framework of a first-order Bethe lattice, we investigate the phase transitions and critical phenomena exhibited by this model. The Bethe lattice provides a simplified yet powerful structure for studying phase transitions in systems with large coordination numbers.

Introduction:

A Bethe lattice Γ^k (or Cayley tree in another terminology, see [1] for the subject of terminology) of order k ≥ 1 is an infiniΓte tree, i.e., a graph without cycles, from each vertex of which precisely k + 1 lines emanate. Let Γ^k =(V,L,i), i.e., V is the set of vertices of Γ^k, L is the set of its lines, and i is the incidence function, which associates with each line $l \in L$ its end points $x,y \in V$. If i(l)= {x, y}, then x and y are said to be neighboring vertices, and in this case we shall write l = <x,y>.

The distance $d(x,y)$. $x,y \in$ V, is introduced on the Bethe lattice in accordance with the formula d(x,y)=min$\{d/\exists = x_0, x_1, x_2 \dots x_{d-1}\ x_d = y \in V$ such that the pairs <x0, x1> , ..., $<x_{d-1}, x_d>$ {are nearest neighbors}. A sequence $\pi=\{x_0, x_1, x_2 \dots \dots x_{d-1}, x_d = y \in V\}$ *that* realizes this minimum is called a path from x to y. In the Potts model on a Bethe lattice, the spin variables σ(x), $x \in V$ take the values 1,2, ..., q, and the Hamiltonian has the form

$$H(\sigma) = -J_1 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} - J_2 \sum_{>x,y<} \delta_{\sigma(x)\sigma(y)}$$

where δ is the Kronecker delta and the summation is over all pairs of nearest neighbors.

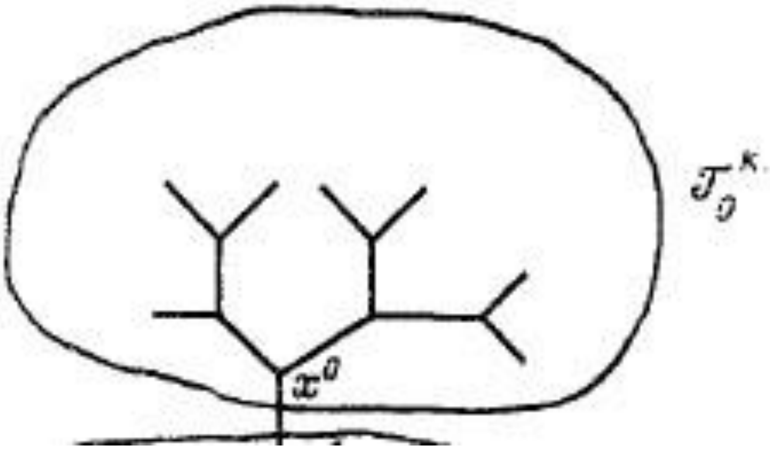
The ferromagnetic Potts model is determined by the Hamiltonian (1.1) for J > 0. In this paper, we restrict ourselves to studying the ferromagnetic Potts model with zero external field, i.e., H = 0. The concepts of Gibbs distribution of the Ports model on the Bethe lattice, limiting Gibbs distribution, and pure phase (extreme Gibbs distribution) are introduced in the usual manner (see [2-9]). The content of the paper is as follows. In Sec. 2, we derive recursion relations and study the fixed points of the transformations corresponding to them; in Sec. 3, we construct pure phases. In this paper, we restrict ourselves to the case q = 3 and k = 2, doing this solely for the relative simplicity of the calculations made below.

Ports model on the Bethe lattice:

Suppose the set of values of the spin variables σ(x), $x \in V$, consists of vectors $\sigma_i \in R$ 6R 2, i=1, 2, 3,4,5 such that $|\sigma_i| = 1$ for all i .

the Hamiltonian (1.1) with external field H = 0 reduces to

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y)$$



with the description of the general structure of limiting Gibbs distributions on a Bethe lattice for models with Hamiltonian of the form (2.1). The proofs of these theorems can be found in [3,10,11,12].

If from a Bethe lattice Γ^k we remove an arbitrary line

$<x^0, x^1> = l \in L$, then it breaks up

into two components -- two semi-infinite Bethe lattices Γ_0^k and Γ_1^k (see Fig. 1, where **k = 1).**

THEOREM 2.1. A necessary and sufficient condition for V to be a limiting Gibbs distribution on Γ^k is that there exist limiting Gibbs distribution μ_0 and μ_1 (which are uniquely determined) on Γ_0^k, Γ_1^k respectively, such that

$$\mu = \frac{\mu_0 \mu_1 \exp\left\{\left(\frac{J}{T}\right) \sigma(x^0)\sigma(x^1)\right\}}{Z},$$

where Z > 0 is a normalization constant

Theorem 2.1 reduces the description of limiting Gibbs distributions on Γ^k to a semiinfinite lattice Γ_0^k . Let V^0 be the set of vertices of Γ_0^k and L^0 its set of lines. On Γ_0^k there is a distinguished boundary vertex $x^0 \in V^0$ from which k lines emanate. We shall call the set. The Gibbs distributions ∈Γ possess the following simple but important property.

THEOREM 2.3. Let ∈Γ and $\{\mu_x, x \in V^0\}$ be corresponding Gibbs distributions on $\{\Gamma_x^k, x \in V^0\}$.

Then for any $x \in V^0$ $\mu_x, \in \Gamma(\Gamma_x^k)$, and the corresponding Gibbs distribution that ensure the factorization property (2.3) are $\{\mu_x, y \geq x\}$.

Let ∈Γ and $\{\mu_x, x \in V^0\}$ be the Gibbs distributions corresponding to on $\{\Gamma_x^k, x \in V^0\}$ We consider the distribution of the spin σ(x)= σ_i, i=1, 2, 3, 4, 5, with respect to μ_x and write it in the form , $p_i = \mu_x(\sigma(x) = \sigma_i)$, i=1,2,3,4,5. We show that there exists a unique vector h∈R^2 such that

$P_i = \frac{exp(h\sigma_i)}{\sum_{j=1}^{3} \exp(h\sigma_i)}$, i=1,2,3,4,5

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y)$$

Results:

$$\begin{cases} \phi = \{1,2, \dots . q\} \\ \Gamma = (V, L, i) \\ \sigma: V \to \phi \\ x \in V \to \sigma(x) \in \{\phi\} \end{cases}$$

$$H(\sigma) = -J_1 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} - J_2 \sum_{>x,y<} \delta_{\sigma(x)\sigma(y)}$$

where $J_1 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)}$ is nearest neighbor

$J_2 \sum_{>x,y<} \delta_{\sigma(x)\sigma(y)}$ is next nearest neighbor

$\Lambda_n = \{0.1.2 \dots . n\}$ $\Lambda_n^c = Z_+$

$\sigma_n: \Lambda_n \to \Phi$ $\overline{\sigma^c}: \Lambda_n^c \to \Phi$ $| \Omega_n | = q^{n+1}$

$H(\sigma/\overline{\sigma^n}) = -J \sum_{<x,y>} \delta_{\sigma(x)_n \sigma(y)_n} - J_1 \delta_{\sigma(x)_n \overline{\sigma^n}(y)}$

$\mu_n(\sigma \mid \overline{\sigma^n}) = \frac{\exp(-\beta H(\sigma|\overline{\sigma^n})}{Z_n(\overline{\sigma^n})}$

$Z_n = \sum_{\sigma_n \in \Omega_n} \exp(-\beta H(\sigma \mid \overline{\sigma^n}))$

$\Omega_n = \Omega_n^1 \cup \Omega_n^2 \cup \dots \cup \Omega_n^q$, $\Omega_n^i = \{\sigma_n \in \Omega_n, \ \sigma_n(0) = i\}$, $| \Omega_n^i | = q^n$

$Z_n^i = \sum_{\sigma_n \in \Omega_n^i} \exp(-\beta H(\sigma \mid \overline{\sigma^n})$, $Z_n = \sum_i^q Z_n^i$

q=4 $Z_n^1, Z_n^2, Z_n^3, , Z_n^4$ $U_n = \frac{Z_n^1}{Z_n^4}$ $V_n = \frac{Z_n^2}{Z_n^4}$ $K_n = \frac{Z_n^3}{Z_n^4}$

$Z_{n+1}^1 = \theta Z_n^1 + Z_n^2 + Z_n^3 + Z_n^4$, $Z_{n+1}^2 = Z_n^1 + \theta Z_n^2 + Z_n^3 + Z_n^4$,

$Z_{n+1}^3 = Z_n^1 + Z_n^2 + \theta Z_n^3 + Z_n^4$ $Z_{n+1}^4 = Z_n^1 + Z_n^2 + Z_n^3 + \theta Z_n^4$

$U_n' = \frac{Z_{n+1}^1}{Z_{n+1}^4} = \frac{\theta Z_n^1 + Z_n^2 + Z_n^3}{Z_n^1 + Z_n^2 + Z_n^3 + \theta Z_n^4}$ $V_n' = \frac{Z_{n+1}^2}{Z_{n+1}^4} = \frac{Z_n^1 + \theta Z_n^2 + Z_n^3}{Z_n^1 + Z_n^2 + Z_n^3 + \theta Z_n^4}$ $K_n' = \frac{Z_{n+1}^3}{Z_{n+1}^4} = \frac{Z_n^1 + Z_n^2 + \theta Z_n^3}{Z_n^1 + Z_n^2 + Z_n^3 + \theta Z_n^4}$

$$\begin{cases} \lim_{n\to\infty} U_n = U & \lim_{n\to\infty} U_n' = U \\ \lim_{n\to\infty} V_n = V & \lim_{n\to\infty} V_n' = U \\ \lim_{n\to\infty} K_n' = K & \lim_{n\to\infty} K_n' = K \end{cases}$$

$$U = \frac{\theta U + V + K + 1}{U + V + K + \theta} \quad V = \frac{U + \theta V + K + 1}{U + V + K + \theta} \quad K = \frac{U + V + \theta K + 1}{U + V + K + \theta}$$

U+V+K=t

U+V+K=$\frac{\theta U+V+K+1}{U+V+K+\theta} + \frac{U+\theta V+K+1}{U+V+K+\theta} + \frac{U+V+\theta K+1}{U+V+K+\theta} = \frac{(\theta+2)(U+V+K)+3}{U+V+K+\theta}$

t=$\frac{(\theta+2)t+3}{t+\theta}$ $\to$ $t^2 - 2t - 3 = 0$ $t_1 = 3$, $t_2 = -1$

V+K=3-U $U=\frac{\theta U+3-U+1}{U+3-U+\theta}$

3U=4-U; U=1

V+K=2 => K=2-V $V=\frac{1+\theta V+2-V+1}{1+V+2-V+\theta}$

3V=4-V => V=1 K=2-1 => K=1

U=1, V=1 , K=1 q =5 $Z_n^1, Z_n^2, Z_n^3,, Z_n^4, Z_n^5$

$U_n=\frac{Z_n^1}{Z_n^5}$ $V_n=\frac{Z_n^2}{Z_n^5}$ $K_n=\frac{Z_n^3}{Z_n^5}$ $L_n=\frac{Z_n^4}{Z_n^5}$

$Z_{n+1}^1=\theta Z_n^1+Z_n^2+Z_n^3+Z_n^4+Z_n^5$, $Z_{n+1}^2=Z_n^1+\theta Z_n^2+Z_n^3+Z_n^4+Z_n^5,$

$Z_{n+1}^3=Z_n^1+Z_n^2+\theta Z_n^3+Z_n^4+Z_n^5$ $Z_{n+1}^4=Z_n^1+Z_n^2+Z_n^3+\theta Z_n^4+Z_n^5$

$Z_{n+1}^5=Z_n^1+Z_n^2+Z_n^3+Z_n^4+\theta Z_n^5$

$U_n'=\frac{Z_{n+1}^1}{Z_{n+1}^5}=\frac{\theta Z_n^1+Z_n^2+Z_n^3+Z_n^4+Z_n^5}{Z_n^1+Z_n^2+Z_n^3+Z_n^4+\theta Z_n^5}$ $V_n'=\frac{Z_{n+1}^2}{Z_{n+1}^5}=\frac{Z_n^1+\theta Z_n^2+Z_n^3+Z_n^4+Z_n^5}{Z_n^1+Z_n^2+Z_n^3+Z_n^4+\theta Z_n^5}$ $K_n'=\frac{Z_{n+1}^3}{Z_{n+1}^5}=\frac{Z_n^1+Z_n^2+\theta Z_n^3+Z_n^4+Z_n^5}{Z_n^1+Z_n^2+Z_n^3+Z_n^4+\theta Z_n^5}$

$L_n'=\frac{Z_{n+1}^4}{Z_{n+1}^5}=\frac{Z_n^1+Z_n^2+Z_n^3+\theta Z_n^4+Z_n^5}{Z_n^1+Z_n^2+Z_n^3+Z_n^4+\theta Z_n^5}$

$\lim_{n\to\infty} U_n=U$ $\lim_{n\to\infty} U_n'=U$

$\lim_{n\to\infty} V_n=V$ $\lim_{n\to\infty} V_n'=U$

$\lim_{n\to\infty} K_n=K$ $\lim_{n\to\infty} K_n'=K$

$\lim_{n\to\infty} L_n=L$ $\lim_{n\to\infty} L_n'=L$

$U=\frac{\theta U+V+K+L+1}{U+V+K+L+\theta}$, $V=\frac{U+\theta V+K+L+1}{U+V+K+L+\theta}$, $K=\frac{U+V+\theta K+L+1}{U+V+K+L+\theta}$, $L=\frac{U+V+K+\theta L+1}{U+V+K+L+\theta}$

U+V+K+L=$\frac{\theta U+V+K+L+1}{U+V+K+L+\theta}+\frac{U+\theta V+K+L+1}{U+V+K+L+\theta}+\frac{U+V+\theta K+L+1}{U+V+K+L+\theta}+\frac{U+V+K+\theta L+1}{U+V+K+L+\theta}$

U+V+K+L=$\frac{(\theta+3)(U+V+K+L)+4}{U+V+K+L+\theta}$

U+V+K+L=a

a =$\frac{(\theta+3)a+4}{a+\theta}$ => $a^2-3t-4=0$ => $a_1=-4$, $a_2=1$

U+V+K+L=1 V+K+L=1-U

$$U=\frac{\theta U+1-U+1}{U+1-U+\theta}$$

U- $\theta U=\theta U+2-U,$ 2U=2 , U=1

U+K+L=1-V $V=\frac{\theta V+1-V+1}{1-V+V+\theta}$

V+ $\theta V=\theta V+2-V,$ 2V=2 , V=1,

U+V+L=1-K

$$K=\frac{\theta K+1-K+1}{1-K+K+\theta}$$

K+ $\theta K=\theta K+2-K,$ 2K=2, K=1

U+V+K=1-L

$$L=\frac{1-L+\theta L+1}{1-L+L+\theta}$$

L- $\theta L=\theta L+2$, 2L=2 , L=1

Conclusion. Since U=V=K=L=1, one can conclude that the limit Gibbs state is the uniform distribution.

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