ω-LIE ALGEBRAS AND THEIR CLASSIFICATION

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https://doi.org/10.5281/zenodo.17036660

ARTICLE INFO

Qabul qilindi: 25-Avgust 2025 yil
Ma'qullandi: 28- Avgust 2025 yil
Nashr qilindi: 31- Avgust 2025 yil

KEYWORDS

ω-Lie algebra, Jordan canonic form, bilinear, Jacobi identity

ABSTRACT

I give the algebraic classification of complex four-dimensional ω-Lie algebras.

MSC2020: 17B99 (primary); 17B30, 17B56 (secondary)

Let F be a field of characteristic zero and L be a finite-dimensional vector space over F. Let $[-.-]: L \times L \to L$ be an anti-commutative product on L and $\omega: L \times L \to k$ be a skew-symmetric bilinear form on L. The triple $(L, [-.-], \omega)$ is called an *ω-Lie algebra* if the following condition is satisfied:

$$[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = \omega(e_1, e_2)e_3 + \omega(e_2, e_3)e_1 + \omega(e_3, e_1)e_2 \quad (1)$$

for all $e_1, e_2, e_3 \in L$. The equation (1) is called the *ω-Jacobi identity*. Apparently, the ω is also skew-symmetric; an ω-Lie algebra is a Lie algebra if and only if the bilinear form $\omega \equiv 0$. So we usually call the Lie algebras *trivial* ω-Lie algebras.

The concept of ω-Lie algebras, connected to the investigation of isoparametric hypersurfaces in Riemannian geometry ([1, 4]), was presented in the recent work of Nurowski [3]. From the definition, it is apparent that all ω-Lie algebras are trivial in dimensions 1 and 2. The initial instance of a non-trivial 3-dimensional ω-Lie algebra was provided in [3]. In that article, Nurowski ultimately finalized the classification of 3-dimensional ω-Lie algebras over the field of real numbers.

A fundamental advancement in the theory of ω-Lie algebras was made by Zusmanovich [5], who introduced numerous basic concepts, such as modules, (quasi-) ideals, and (generalized) derivations; he also discovered several fundamental properties of these algebras. One of Zusmanovich's findings claims that finite-dimensional non-trivial ω-Lie algebras are either low-dimensional or possess an abelian subalgebra of small codimension that satisfies certain restrictive conditions. Specifically, the following practical result is demonstrated.

Lemma 1 *If L is a finite-dimensional ω-Lie algebra with non-degenerate ω, then $\dim L = 2$.*

Proof. Given that ω is nondegenerate, the dimension of L is equal to the rank of ω and is therefore an even number. We will first address the scenario where the dimension of L is greater than or equal to six. To handle this case, we will employ coordinate notation. While this approach may be somewhat less elegant, it will facilitate and simplify the ensuing computations..

The Lie algebra L can be decomposed into the direct sum of two maximal isotropic subspaces, A and B, each possessing a dimension of $n = \frac{\dim L}{2} \geq 3$. It is possible to select a basis $\{a_1, \dots, a_n\}$ for A and a basis $\{b_1, \dots, b_n\}$ for B such that the bilinear form satisfies $\omega(e_i, e_j) = 1$ and $\omega(e_i, e_j) = 0$ if $i \neq j$. Subsequently, according to the established lemmata, each of these isotropic subspaces is either an abelian or an almost abelian Lie subalgebra. Furthermore, as a consequence of the reasoning presented in the lemma's proof, the multiplicative operations within them can be expressed as follows: $[a_i, a_j] = \alpha_j a_i - \alpha_i a_j$ and $[b_i, b_j] = \beta_j b_i - \beta_i b_j$ for some $\alpha_i, \beta_i \in K$. Again, $[a_i, b_j] = \lambda_{ij} a_i + \mu_{ij} b_j$ if $i \neq j$, for some $\lambda_{ij}, \mu_{ij} \in K$.

Writing for elements b_i, b_j, b_k, a_i, where i, j, k are pairwise distinct, and collecting coefficients of b_j, we get **equation** $\omega([a_i, b_i], b_k) = \lambda_{ik}$ for any $i \neq k$. writing for elements a_i, a_j, b_i, b_k, where i, j, k are pairwise distinct, and collecting coefficients of a_j, we get

$$\omega([a_i, b_i], b_k) = \lambda_{ik} - \lambda_{jk} + \beta_k.$$

A comparison of these two equivalent expressions reveals that $\lambda_{jk} = \beta_k$ for all $j \neq k$. Through an entirely analogous line of reasoning, we also obtain.

$$\omega([a_i, b_i], a_k) = -\mu_{ki} = \alpha_k \qquad (2)$$

for any $i \neq k$.

(2) give all coefficients in the decomposition of $[a_i, b_i]$ by elements of the symplectic basis $\{a_1, \dots, a_n, b_1, \dots, b_n\}$, except those of a_i, b_i, so for any $1 \leq i \leq n$ we may write

$$[a_i, b_i] = \sum_{\substack{1 \leq k \leq n \\ k \neq i}} (\beta_k a_k - \alpha_k b_k) + \lambda_i a_i + \mu_i b_i$$

for some $\lambda_i, \mu_i \in K$.

By expressing the product for elements a_i, a_j, b_i, b_j, where $i \neq j$, incorporating all previously established multiplication formulas between elements of A and B, and then grouping the coefficients for a_i and a_j, we arrive at the respective equations: $\lambda_i = 2\beta_i$ and $\lambda_j = -2\beta_j$. It follows that $\lambda_i = \beta_i = 0$ for any $1 \leq i \leq n$. In a similar manner, by grouping the coefficients of b_i and b_j, we get $\mu_i = \alpha_i = 0$.

Consequently, the algebra L is abelian. However, in this case, the ω-Jacobi identity dictates that for any three linearly independent elements, the value of ω on the product of any two must be zero. This ultimately implies that ω itself is the zero form, which contradicts the initial assumption of its nondegeneracy.

The scenario where $dim\ L = 4$ necessitates somewhat more involved computations. It is important to note that we can assume the base field to be algebraically closed, as the nondegeneracy of ω is clearly preserved under any field extension.

Lemma 1 mentioned above means that the bilinear form ω on any ω-Lie algebra L must be degenerate if $\dim L \geq 3$. Our terminology concerning bilinear forms is standard. Let ω be a skew-symmetric bilinear form on a linear space L. A subspace $W \subseteq L$ is called *isotropic* if

$\omega(W,W)=0$. Let $W^{\perp}=\{e_1\in L|\omega(e_1,W)=0\}$ denote the orthogonal complement to a subspace W. Let $Ker\ \omega=L^{\perp}$ denote the kernel of ω.

In what follows, $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers respectively, and L denotes an ω-Lie algebra with a basis $\{e_1,e_2,e_3\}$. We write $\wedge^2 L$ for the exterior power of L with the basis $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$ and $\varphi=[-,-]:\wedge^2 L\to L$ is the bracket product. We use $L'=[L,L]$ to denote the commutator subalgebra of L. We call the dimension of L' the *rank* of φ.

From this Jacobi identity (1), we obtain the following conclusion.

Theorem 1 *Let L be n dimension ω-Lie algebra and dimension* $dimL'=dim[L,L]\le n-2$*, then all of the* $e_i,e_j\in L$*,* $\omega(e_i,e_j)=0$.

Proof. We choose $\{e_i,e_j,e_k\}$ as a linear independent basis of L, and $\{e_1,e_2,\dots,e_{n-2}\}\in L'$ so we can write $[e_i,e_j]=a_{(ij)1}e_1+a_{(ij)2}e_2+a_{(ij)3}e_3+\dots a_{(ij)(n-3)}e_{n-3}+a_{(ij)(n-2)}e_{n-2}$, $i,j,k\ \in N$ and calculate Jacoby identity (1)

$$\omega(e_i,e_j)e_k+\omega(e_j,e_k)e_i+\omega(e_k,e_i)e_j=\big[[e_i,e_j],e_k\big]+\big[[e_j,e_k],e_i\big]+\big[[e_k,e_i],e_j\big]=$$
$$=\left[a_{(ij)1}e_1+a_{(ij)2}e_2+a_{(ij)3}e_3+\cdots a_{(ij)(n-3)}e_{n-3}+a_{(ij)(n-2)}e_{n-2},e_k\right]$$
$$+\left[a_{(jk)1}e_1+a_{(jk)2}e_2+a_{(jk)3}e_3+\cdots a_{(jk)(n-3)}e_{n-3}+a_{(jk)(n-2)}e_{n-2},e_i\right]$$
$$++\left[a_{(ki)1}e_1+a_{(ki)2}e_2+a_{(ki)3}e_3+\cdots a_{(ki)(n-3)}e_{n-3}+a_{(ki)(n-2)}e_{n-2},e_k\right]=$$
$$=a_{(ij)1}(a_{(1k)1}e_1+a_{(1k)2}e_2+a_{(1k)3}e_3+\dots a_{(1k)(n-3)}e_{n-3}+a_{(1k)(n-2)}e_{n-2})$$
$$++a_{(ij)2}(a_{(2k)1}e_1+a_{(2k)2}e_2+a_{(2k)3}e_3+\dots a_{(2k)(n-3)}e_{n-3}$$
$$+a_{(2k)(n-2)}e_{n-2})+\dots++a_{(ij)(n-2)}(a_{((n-2)k)1}e_1+a_{((n-2)k)2}e_2$$
$$+a_{((n-2)k)3}e_3+\dots a_{((n-2)k)(n-3)}e_{n-3}+a_{((n-2)k)(n-2)}e_{n-2})+a_{(ki)1}(a_{(1j)1}e_1$$
$$+a_{(1j)2}e_2+a_{(1j)3}e_3+\dots a_{(1j)(n-3)}e_{n-3}+a_{(1j)(n-2)}e_{n-2})++a_{(ki)2}(a_{(2j)1}e_1$$
$$+a_{(2j)2}e_2+a_{(2j)3}e_3+\dots a_{(2j)(n-3)}e_{n-3}+a_{(2j)(n-2)}e_{n-2})+\dots+$$
$$+a_{(ki)(n-2)}(a_{((n-2)j)1}e_1+a_{((n-2)j)2}e_2$$
$$+a_{((n-2)j)3}e_3+\dots a_{((n-2)j)(n-3)}e_{n-3}+a_{((n-2)j)(n-2)}e_{n-2})+a_{(ki)1}(a_{(1i)1}e_1+a_{(1i)2}e_2$$
$$+a_{(1i)3}e_3+\dots a_{(1i)(n-3)}e_{n-3}+a_{(1i)(n-2)}e_{n-2})++a_{(ki)2}(a_{(2i)1}e_1+a_{(2i)2}e_2$$
$$+a_{(2i)3}e_3+\dots a_{(2i)(n-3)}e_{n-3}+a_{(2i)(n-2)}e_{n-2})+\dots++a_{(ki)(n-2)}(a_{((n-2)i)1}e_1$$
$$+a_{((n-2)i)2}e_2+a_{((n-2)i)3}e_3+\dots a_{((n-2)i)(n-3)}e_{n-3}+a_{((n-2)i)(n-2)}e_{n-2})=$$
$$=(a_{(ij)1}a_{(1k)1}+a_{(ij)2}a_{(2k)1}+\dots+a_{(ij)(n-2)}a_{((n-2)k)1}+a_{(ki)1}a_{(1j)1}$$
$$+a_{(ki)2}a_{(2j)1}+\dots+a_{(ki)(n-2)}a_{((n-2)j)1}+a_{(jk)1}a_{(1i)1}$$
$$+a_{(jk)2}a_{(2i)1}+\dots+a_{(jk)(n-2)}a_{((n-2)i)1})e_1++(a_{(ij)1}a_{(1k)2}$$
$$+a_{(ij)2}a_{(2k)2}+\dots+a_{(ij)(n-2)}a_{((n-2)k)2}+a_{(ki)1}a_{(1j)2}$$
$$+a_{(ki)2}a_{(2j)2}+\dots+a_{(ki)(n-2)}a_{((n-2)j)2}+a_{(jk)1}a_{(1i)2}$$
$$+a_{(jk)2}a_{(2i)2}+\dots+a_{(jk)(n-2)}a_{((n-2)i)2})e_2+\dots++(a_{(ij)1}a_{(1k)(n-2)}$$
$$+a_{(ij)2}a_{(2k)(n-2)}+\dots+a_{(ij)(n-2)}a_{((n-2)k)(n-2)}+a_{(ki)1}a_{(1j)(n-2)}$$
$$+a_{(ki)2}a_{(2j)(n-2)}+\dots+a_{(ki)(n-2)}a_{((n-2)j)(n-2)}+a_{(jk)1}a_{(1i)(n-2)}$$
$$+a_{(jk)2}a_{(2i)(n-2)}+\dots+a_{(jk)(n-2)}a_{((n-2)i)(n-2)})e_{n-2}$$

In this case, $e_1,e_2,\dots,e_{n-2}$ form a linearly independent basis, and from the results derived from the Jacobi identity, the following equality is obtained.

From this Jacobi identity (1), we obtain the following conclusion.

We define the coefficients as follows

$\omega(e_i,e_j) = a_{(ij)1}a_{(1k)k} + a_{(ij)2}a_{(2k)k} + \ldots + a_{(ij)(n-2)}a_{((n-2)k)k} + a_{(ki)1}a_{(1j)k} + a_{(ki)2}a_{(2j)k} + \ldots + a_{(ki)(n-2)}a_{((n-2)j)k} + a_{(jk)1}a_{(1i)k} + a_{(jk)2}a_{(2i)k} + \ldots + a_{(jk)(n-2)}a_{((n-2)i)k}$
$\omega(e_j,e_k) = a_{(ij)1}a_{(1k)i} + a_{(ij)2}a_{(2k)i} + \ldots + a_{(ij)(n-2)}a_{((n-2)k)i} + a_{(ki)1}a_{(1j)i} + a_{(ki)2}a_{(2j)i} + \ldots + a_{(ki)(n-2)}a_{((n-2)j)i} + a_{(jk)1}a_{(1i)i} + a_{(jk)2}a_{(2i)i} + \ldots + a_{(jk)(n-2)}a_{((n-2)i)i}$
$\omega(e_k,e_i) = a_{(ij)1}a_{(1k)j} + a_{(ij)2}a_{(2k)j} + \ldots + a_{(ij)(n-2)}a_{((n-2)k)j} + a_{(ki)1}a_{(1j)j} + a_{(ki)2}a_{(2j)j} + \ldots + a_{(ki)(n-2)}a_{((n-2)j)j} + a_{(jk)1}a_{(1i)j} + a_{(jk)2}a_{(2i)j} + \ldots + a_{(jk)(n-2)}a_{((n-2)i)j}$
$a_{(ij)1}a_{(1k)1} + a_{(ij)2}a_{(2k)1} + \ldots + a_{(ij)(n-2)}a_{((n-2)k)1} + a_{(ki)1}a_{(1j)1} + a_{(ki)2}a_{(2j)1} + \ldots + + a_{(ki)(n-2)}a_{((n-2)j)1} + a_{(jk)1}a_{(1i)1} + a_{(jk)2}a_{(2i)1} + \ldots + a_{(jk)(n-2)}a_{((n-2)i)1} = 0$
....
$a_{(ij)1}a_{(1k)(n-2)} + a_{(ij)2}a_{(2k)(n-2)} + \ldots + a_{(ij)(n-2)}a_{((n-2)k)(n-2)} + a_{(ki)1}a_{(1j)(n-2)} + a_{(ki)2}a_{(2j)(n-2)} + \ldots + a_{(ki)(n-2)}a_{((n-2)j)(n-2)} + a_{(jk)1}a_{(1i)(n-2)} + a_{(jk)2}a_{(2i)(n-2)} + \ldots + a_{(jk)(n-2)}a_{((n-2)i)(n-2)} = 0$

In this case, the main concept to be demonstrated is that $\omega(e_i,e_j) = 0$ for arbitrary e_i and e_j, where $e_1, e_2, \ldots, e_{n-2} \in L'$ and $e_{n-1}, e_n \notin L'$. From this, it follows that when we consider the Jacobi identity involving e_i, e_j, and e_n, the result $\omega(e_i,e_j) = 0$ is obtained, similar to the case when the full n-dimensional Jacobi identity is considered. However, the expressions $\omega(e_n,e_j)$ and $\omega(e_i,e_n)$ may be nonzero. In the n-dimensional case, the total number of Jacobi identities to be considered is $\binom{n}{3}$. When we consider the Jacobi identity for the triple e_i, e_n, e_{n-1}, it turns out that $\omega(e_i,e_n) = 0$, and similarly, $\omega(e_n,e_j) = 0$ is obtained.

In this situation, $\omega(e_n,e_{n-1}) = 0$ also holds, since the Jacobi identity for the triple e_n, e_{n-1}, e_n must also be satisfied.

From this theorem, we derive the following important lemma.

Lemma 2 *When classifying any n-dimensional ω-Lie algebras, it is sufficient to consider the cases when the rank of φ (i.e., the dimension of L') is $n-1$ and n.*

Example 1. Let us consider a four-dimensional ω-Lie algebra. In studying its classification, it suffices to consider the cases where the rank of the associated bilinear form is 3 or 4, since in other cases it reduces to a trivial Lie algebra

Theorem 3.1.1 Let L be a nontrivial (i.e. non-Lie) 4-dimensional ω-Lie algebra, then it must be isomorphic to one of the following algebras:

Table 1: Nontrivial Complex Four-dimensional ω-Lie Algebras

type of Algebra	Commutation relations
A_1	$[e_1,e_3] = -e_1$, $[e_2,e_3] = -e_2$, $[e_4,e_1] = e_1$, $[e_4,e_2] = e_1 + e_2$, $\omega(e_4,e_3) = -1$
$L_{a,b}$	$[e_3,e_1] = e_1$, $[e_3,e_2] = e_2$, $[e_4,e_1] = be_1$, $= ae_1 + be_2$, $[e_4,e_3] = e_3$, $\omega(e_4,e_3) = 1$.
L^1	$[e_1,e_2] = e_1$, $[e_3,e_2] = e_3$, $[e_4,e_2] = e_1$, $[e_4,e_3] = e_2, \omega(e_4,e_3) = -1$
A_2	$[e_1,e_3] = e_2$, $[e_1,e_4] = e_3$, $[e_2,e_3] = e_1, \omega(e_4,e_2) = 1$.

L_c	$[e_1,e_2]=e_1,\ [e_1,e_3]=e_1+e_2,\ [e_2,e_3]=-ce_1+e_3,$ $[e_3,e_4]=-e_4,\ \omega(e_4,e_2)=c, where c\in\mathbb{C}.$
$C_{a,b}$:	$[e_1,e_2]=3e_1,\quad [e_1,e_4]=e_3,\quad [e_2,e_3]=e_1-2e_3,$ $[e_2,e_4]=ae_1+be_3+e_4,\ [e_3,e_4]=-0.5e_1-e_3;\ \omega(e_4,e_2)=-1$
D^1:	$[e_1,e_3]=e_3,\quad [e_1,e_4]=e_4,\quad [e_2,e_3]=e_1$ $[e_2,e_4]=e_4,\ \omega(e_3,e_2)=1$
$E_{\alpha,\beta}$:	$[e_1,e_2]=e_2,\ [e_1,e_3]=\alpha e_3,\ [e_1,e_4]=(1-\alpha)e_4,\ [e_2,e_3]=e_4, [e_2,e_4]=e_3,\ [e_3,e_4]=e_1+\beta e_2;$ $\omega(e_3,e_4)=1. where \alpha,\beta\in\mathbb{C}.$
F_γ:	$[e_1,e_2]=2e_2,\quad [e_1,e_3]=e_3,\quad [e_1,e_4]=\gamma e_3+e_4,$ $[e_3,e_4]=e_1,\ \omega(e_3,e_4)=2, where \gamma\in\mathbb{C}.$

Rank 3 In this scenario, we select e_1,e_2,e_3 as the basis of L' with e_4 not in L'. Assuming $[e_1,e_2]=ae_1+by+ce_3,\ [e_3,e_1]=me_1+ne_2+ke_3,$ and $[e_3,e_2]=pe_1+qe_2+re_3,$ at least one of $[e_1,e_2]$, $[e_3,e_2]$, or $[e_3,e_1]$ must be zero, as the kernel of φ is one-dimensional.

To establish the bijectiveness of the linear map ad $_{e_4}: L'\to L'$ defined by $u\mapsto[e_4,u]$ we can leverage the properties of Lie algebras. This map is related to the adjoint representation, where ad $_{e_4}$ corresponds to the action of the element e_1,e_2,e_3 on the Lie algebra $[L,L]$ through the Lie bracket.

Therefore the linear map ad $_{e_4}: L'\to L'$ by $u\mapsto[e_4,u]$ is bijective. By course of linear algebra, we able to choose the suitable basis elements e_1,e_2,e_3 as that ad $_{e_4}$ is similar to

$$A=\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix},\quad B=\begin{pmatrix}\delta & 1 & 0\\ 0 & \delta & 0\\ 0 & 0 & \psi\end{pmatrix},\quad C=\begin{pmatrix}\mu & 0 & 0\\ 0 & \nu & 0\\ 0 & 0 & \tau\end{pmatrix} \tag{3}$$

where δ,μ,ν and τ are all nonzero and are the eigenvalues of ad $_{e_4}$. Thus our arguments consist of the following four cases.

Case 1 In the first situation, ad $_{e_4}$ is similar to A,

$[e_1,e_2]=ae_1+be_2+ce_3$, $\quad [e_3,e_1]=me_1+ne_2+ke_3$, $\quad [e_3,e_2]=pe_1+qe_2+re_3$,

$ad_{e_4}(e_1)=[e_4,e_1]=\lambda e_1$, $\quad ad_{e_4}(e_2)=[e_4,e_2]=e_1+\lambda e_2,\quad ad_{e_4}(e_3)=[e_4,e_3]=e_2+\lambda e_3$

then one of the $[e_1,e_2],[e_3,e_2],[e_3,e_1]$ because the kernel of φ is one dimensional , we seperate in thus situation 3 subcase.

subcase 1. assume that $[e_1,e_2]=0$

$[e_1,e_2]=0$, $[e_3,e_1]=me_1+ne_2+ke_3$, $[e_3,e_2]=pe_1+qe_2+re_3$, $[e_4,e_1]=\lambda e_1$,
$[e_4,e_2]=e_1+\lambda e_2$ $[e_4,e_3]=e_2+\lambda e_3$

By ω-Jacobi identity, we have

$i)\ \omega(e_1,e_2)e_3+\omega(e_2,e_3)e_1+\omega(e_3,e_1)e_2=$
$=kpe_1+kqe_2+kre_3-rme_1-rne_2-rke_3=(kp-rm)e_1+(kq-rn)e_2$

$\omega(e_1,e_2)=0,\ \omega(e_3,e_1)=kq-rn,\ \omega(e_2,e_3)=kp-rm.$

$ii)\ \omega(e_1,e_2)e_4+\omega(e_2,e_4)e_1+\omega(e_4,e_1)e_2=0$

$\omega(e_1,e_2)=0,\ \omega(e_2,e_4)=0,\ \omega(e_4,e_1)=0.$

$iii)\ \omega(e_1,e_3)e_4+\omega(e_3,e_4)e_1+\omega(e_4,e_1)e_3=$
$=\lambda me_1+ne_1+\lambda ne_2+ke_2+\lambda ke_3-\lambda me_1-\lambda ne_2-\lambda ke_3-\lambda me_1-\lambda ne_2-\lambda ke_3=$
$=(n-\lambda m)e_1+(k-\lambda n)e_2-\lambda ke_3$
$\omega(e_1,e_3)=0,\ \omega(e_3,e_4)=n-\lambda m,\ \omega(e_4,e_1)=k-\lambda n\ and\ \lambda k=0.$
$iv)\ \omega(e_2,e_3)e_4+\omega(e_3,e_4)e_2+\omega(e_4,e_2)e_3=$
$=\lambda pe_1+qe_1+\lambda qe_2+re_2+\lambda re_3-\lambda pe_1-\lambda qe_2-\lambda re_3-me_1-ne_2--\lambda re_3-$
$ke_3-\lambda pe_1-\lambda qe_2=(q-\lambda p-m)e_1+(r-\lambda q-n)e_2-(\lambda r+k)e_3$
$\omega(e_3,e_2)=0,\ \omega(e_3,e_4)=r-\lambda q-n,\ \omega(e_4,e_2)=-\lambda r-k\ and\ q-\lambda p-m=0.$
so, find between coefficient

$kp-mr=0$	$q=\lambda p+m$	$n-\lambda m=\omega(e_3,e_4)=r-\lambda q-n$
$kq-nr=0$	$k=\lambda$ n	$\lambda k=0$
$\lambda r+k=\omega(e_2,e_4)=0$		

$$r=0,\qquad k=0,\qquad m=q,\qquad p=0,\qquad n=0$$

Since $m,\lambda\neq 0$ because if $m,\lambda=0$, than we face Lie algebra.
If we take a new basis we obtain the following:

$$x=e_1,\qquad y=e_2,\qquad z=\frac{e_3}{m},\qquad and\ \ t=\frac{e_4+\frac{e_2}{m}}{\lambda}$$

Rank 3

Case 1. If ad $_{e_4}$ is similar to A, then then after computing the ω-Jacobi identity, we obtain the following nontrivial ω Lie algebra.

A_1: $[e_1,e_2]=0,[e_1,e_3]=-e_1,[e_2,e_3]=-e_2,[e_4,e_1]=e_1,$
$[e_4,e_2]=e_1+e_2,[e_4,e_3]=e_3,\omega(e_4,e_3)=-1, where\alpha\in\mathbb{C}.$

Case 2. If ad $_{e_4}$ is similar to B, then then after computing the ω-Jacobi identity, we obtain the following nontrivial ω Lie algebra.

L_{ab}: $[e_1,e_3]=-e_1,[e_2,e_3]=-e_2,[e_4,e_1]=be_1,[e_4,e_2]=e_1+be_2,$
$[e_4,e_3]=ae_3,\omega(e_4,e_3)=a, wherea,b\in\mathbb{C}.$

Rank 4 Next we consider the nontrivial case. Since the dimension of L' is 4, the rank of adjoint map ad $_x: L\to L$ must be 3 for any nonzero $x\in L$. Thus the kernel of ad $_x$ is equal to Cx.

By Lemma if ω is non-degenerate, then L must have dimension 2. So the bilinear form ω we consider here is degenerate. This means that there exists an nonzero element $x\in L$ such that $\omega(x,v)=0$ for all $v\in L$. Now we fix x. By the Jordan canonical form, we can choose a suitable basis $\{u,y,z\}$ of L such that ad $_x$ is similar to

$$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&\mu\end{pmatrix},\quad C=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&\nu&1\\0&0&0&\nu\end{pmatrix}, or$$

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix}, E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \psi & 0 & 0 \\ 0 & 0 & \varepsilon & 1 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}, G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Omega & 1 & 0 \\ 0 & 0 & \Omega & 1 \\ 0 & 0 & 0 & \Omega \end{pmatrix},$$

where δ, μ, ν and τ are all nonzero and are the eigenvalues of ad $_x$. Thus our arguments consist of the following four cases.

Rank 4 If ad $_{e_1}$ is similar to A :

A^2: $[e_1, e_3] = e_2,\ [e_4, e_1] = -e_3,\ [e_4, e_2] = e_1$

$\omega(e_4, e_2) = 1.$

If ad $_{e_1}$ is similar to B :

L_c: $[e_1, e_2] = e_1,\ [e_1, e_3] = e_1 + e_2,\ [e_2, e_3] = -ce_1 + e_3,$

$[e_3, e_4] = -e_4,\ \omega(e_4, e_2) = c, wherec \in \mathbb{C}.$

If ad $_{e_1}$ is similar to C :

$C_{a,b}$: $[e_1, e_2] = 3e_1,\ [e_1, e_4] = e_3,\ [e_2, e_3] = e_1 - 2e_3,$

$[e_2, e_4] = ae_1 + be_3 + e_4,\ [e_3, e_4] = -0.5e_1 - e_3;\ \ \omega(e_4, e_2) = -1$

If ad $_{e_1}$ is similar to D :

D^1: $[e_1, e_3] = e_3,\ [e_1, e_4] = e_4,\ [e_2, e_3] = e_1$

$[e_2, e_4] = e_4,\ \omega(e_3, e_2) = 1$

If ad $_{e_1}$ is similar to E :

$E_{\alpha,\beta}$: $[e_1, e_2] = e_2,\ [e_1, e_3] = \alpha e_3,\ [e_1, e_4] = (1 - \alpha)e_4,\ [e_2, e_3] = e_4, [e_2, e_4] = e_3,\ [e_3, e_4] = e_1 + \beta e_2;$

$\omega(e_3, e_4) = 1. where \alpha, \beta \in \mathbb{C}.$

If ad $_{e_1}$ is similar to F :

F_γ: $[e_1, e_2] = 2e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = \gamma e_3 + e_4,$

$[e_3, e_4] = e_1,\ \omega(e_3, e_4) = 2, where \gamma \in \mathbb{C}.$

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